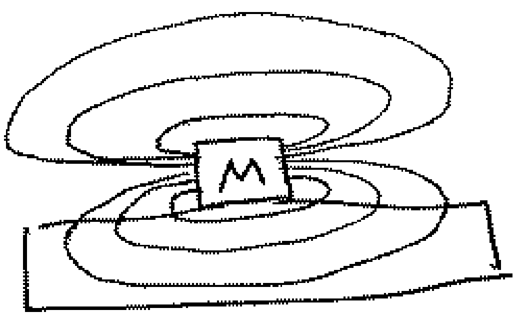
# Physics 472
## Lecture 1

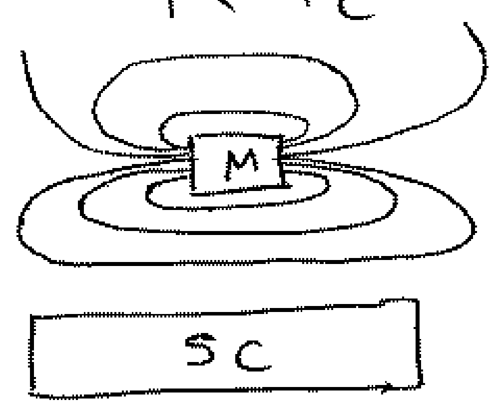
### Meissner effect

$T > T_c$




magnetic field penetrates normal metal

$T < T_c$




magnetic field expelled from superconductor

Q: Can we understand the Meissner effect in terms of perfect conductivity?

Classically, $m \dfrac{d\vec{v}}{dt} = q\vec{E}$

(charged particle of mass $m$ in an electric field)

$$\vec{J}_e = n_s q \langle \vec{v} \rangle$$

$n_s = \frac{\#\text{ carriers}}{\text{volume}}$ $\qquad q = \text{charge}$

$$n_s q \frac{d\langle \vec{v} \rangle}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d\vec{J}_e}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d}{dt} \nabla \times \vec{J}_e = \frac{n_s q^2}{m} \nabla \times \vec{E}$$

But $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

$$\Rightarrow \frac{d}{dt}\left[\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B}\right] = 0$$

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = \text{const.}$$

Currents flow only on the surface

of a perfect conductor, so in order to force $\vec{B} = 0$ in the interior, we must have the integration constant in the above equation $= 0$.

$$\boxed{\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = 0 \qquad \text{London equation}}$$

The London equation describes the Meissner effect. Perfect conductivity <u>does not</u> imply the Meissner effect, since a perfect conductor maintains a <u>constant field</u> in its interior, but this constant field need not be zero!

The Meissner effect is a quantum effect, which occurs in superconductors, but would not occur in a perfect classical conductor.

In order to see why the constant in the London equation is zero, we need to see how the Schrödinger equation is modified in the presence of an (electro)magnetic field:

<u>Maxwell's equations</u>: (cgs units)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 4\pi \rho_e$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} J_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

## Vector and scalar potentials

$\vec{B} = \nabla \times \vec{A}$, $\vec{A}(\vec{r},t) =$ vector potential

$\vec{E} = -\nabla V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $V =$ scalar potential

## Force on a charged particle

(classical):

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

## Gauge invariance:

$$\vec{A}' = \vec{A} + \nabla f(\vec{r},t)$$

$$V' = V - \frac{1}{c}\frac{\partial f}{\partial t}$$

$\vec{B}' = \vec{B}$, $\vec{E}' = \vec{E}$

## Similar symmetry in QM

$$\psi'(\vec{r},t) = e^{i\theta(\vec{r},t)}\,\psi(\vec{r},t)$$

$\rho(\vec{r},t) = |\Psi|^2$ unchanged

But $\vec{P}\Psi' = e^{i\theta}(\vec{P} + \hbar\nabla\theta)\Psi$.

(Recall $\vec{P} = \frac{\hbar}{i}\nabla$ = momentum operator.)

Define $\vec{P}' = \frac{\hbar}{i}\nabla - \hbar\nabla\theta$

Then $\vec{P}'\Psi' = e^{i\theta}\vec{P}\Psi$

$$\vec{J} = Re\left\{\Psi^* \frac{\vec{P}}{m}\Psi\right\} = Re\left\{(\Psi')^* \frac{\vec{P}'}{m}\Psi'\right\}$$

Physical observables (e.g. $\rho$, $\vec{J}$) are unchanged under the transformation

$$\Psi \rightarrow e^{i\theta(\vec{r},t)}\Psi$$

$$\vec{P} \rightarrow \vec{P} - \hbar\nabla\theta$$

# Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + q V(\vec{r},t) \psi$$

$$\psi = e^{-i\theta} \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} + \hbar \frac{\partial \theta}{\partial t} \psi' = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \hbar \nabla \theta \right)^2 \psi' + q V(\vec{r},t) \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \hbar \nabla \theta \right)^2 \psi' + q \left( V - \frac{\hbar}{q} \frac{\partial \theta}{\partial t} \right) \psi'$$

Looks like a gauge transformation with

$$f(\vec{r},t) = \frac{\hbar c}{q} \theta(\vec{r},t)$$

If we introduce the "kinetic momentum"

$$\vec{P}_{kin} = \frac{\hbar}{i} \nabla - \frac{q}{c} \vec{A},$$

then under a gauge transformation

$$\vec{P}'_{kin} = \vec{P}_{kin} - \frac{q}{c}\nabla f = \vec{P}_{kin} - \hbar\nabla\theta \checkmark$$

The gauge-invariant form of Schrödinger's equation is

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}(\vec{r},t)\right)^2\psi + qV(\vec{r},t)\psi}$$

Cf. Classical Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + qV,$$

$$m\vec{v} = \vec{P} - \frac{q}{c}\vec{A}$$

$\vec{p}$ = "canonical momentum"

Quantum mechanically, the electric current is

$$\vec{J}_e = q \, \mathrm{Re}\left\{ \psi^* \vec{v} \, \psi \right\}$$

$$\equiv \frac{q}{m} \mathrm{Re}\left\{ \psi^* \left(\vec{P} - \frac{q}{c}\vec{A}\right) \psi \right\}$$

$$\boxed{\vec{J}_e = \frac{q}{m} \mathrm{Re}\left\{ \psi^* \left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right) \psi \right\}}$$

In a superconductor, the charge carriers condense into a single, macroscopic wave function

$$\psi_s(\vec{r}) = \sqrt{n_s(\vec{r})} \, e^{i\theta(\vec{r})}.$$

$$n_s(\vec{r}) = \left|\psi_s(\vec{r})\right|^2$$

$$\Rightarrow \vec{J}_e = \frac{n_s q}{m}\left(\hbar \nabla\theta - \frac{q}{c}\vec{A}\right)$$

$$\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \nabla \times \vec{A}$$

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = 0 \quad !$$

London equation follows trivially from QM def. of current, provided all carriers are in the same wavefunction $\psi_s(\vec{r})$.

Q: If electrons are fermions, and must obey the Pauli exclusion principle, how can they all have the same wave function in a superconductor?!

A: They don't! The charge

carriers in a SC are "Cooper pairs" of electrons, with charge $q = -2e$ and mass $m = 2m_e$. Such pairs of electrons are bosons, and can condense into a single "ground state" wave function.

## Penetration depth

At the surface of a SC, the currents which screen out magnetic fields from the interior flow, and the magnetic field can penetrate a short distance.

Combining the London equation $\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \vec{B}$ and Ampere's law (for static fields)

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e$$ gives

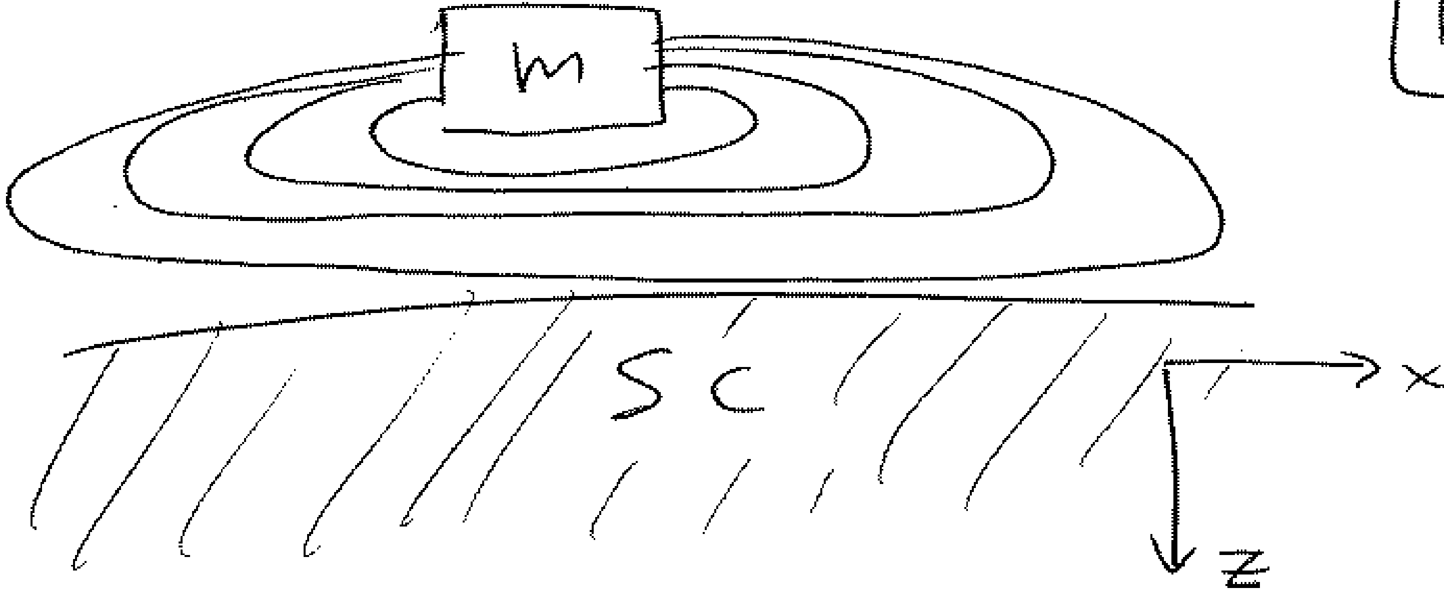
$$\nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B} = \frac{4\pi}{c} \nabla \times \vec{J}_e$$

$$\Rightarrow \quad \nabla^2 \vec{B} = \frac{4\pi n_s q^2}{mc^2} \vec{B}$$

Define the <u>London penetration depth</u>

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_s q^2}}$$

$$\nabla^2 \vec{B} = \lambda_L^{-2} \vec{B}$$




Neglecting edge effects,

$$\vec{B}(\vec{r}) = \hat{x}\, B_x(z)$$

$$\frac{d^2 B_x}{dz^2} = \frac{1}{\lambda_L^2} B_x(z)$$

sol'n: $B_x(z) = B_x(0)\, e^{-z/\lambda_L}$

$\Rightarrow$ Field penetrates exponentially, with a decay length $\lambda_L$.