Physics 472 Lecture 14

The Zeeman effect in Hydrogen

$$H_Z = -\vec{\mu} \cdot \vec{B}_{ext}$$

$$\vec{\mu} = \vec{\mu}_\ell + \vec{\mu}_s$$

$$\vec{\mu}_\ell = -\frac{e}{2m_e c}\vec{L}, \quad \vec{\mu}_s = -\frac{e}{m_e c}\vec{S}$$

$$H_Z = \frac{e}{2m_e c}(\vec{L} + 2\vec{S}) \cdot \vec{B}_{ext}$$

(Here we neglect the Zeeman effect on the proton, which is 3 orders of magnitude smaller.) However, in addition to the external field $\vec{B}_{ext}$, the electron experiences an internal field $\vec{B}_{int}$ due to spin-orbit interaction:

- $$H_{s.o.} = \frac{e^2}{2m_e^2c^2r^3}\vec{L}\cdot\vec{S}$$

$$\equiv -\vec{\mu}_s\cdot\vec{B}_{int}$$

$$\Rightarrow \vec{B}_{int} = \frac{e\vec{L}}{2m_ec\,r^3}$$

- Estimate

$$B_{int} \sim \frac{e\hbar}{2m_ec\,a_0^3}$$

$$= \frac{e}{2a_0^2}\,\frac{\hbar}{m_ec}\,\frac{m_ee^2}{\hbar^2}$$

$$= \frac{e}{2a_0^2}\,\frac{e^2}{\hbar c}$$

- $B_{int} \sim \frac{\alpha}{2}\frac{e}{a_0^2} = 62,600$ gauss $= 6.3$ Tesla

The nature of the Zeeman effect depends critically on the relative size of B_{ext} and B_{int}.

1) Weak-Field Zeeman effect

If $B_{ext} \ll B_{int}$, the spin-orbit interaction dominates, and H_Z can be treated as a perturbation. The "good" quantum #s are n, ℓ, j, m_j (but not m_ℓ or m_s).

$$E_Z^{(1)} = \langle n \ell j m_j | H_Z | n \ell j m_j \rangle$$

$$= \frac{e}{2m_e c} \langle \vec{B}_{ext} \cdot (\vec{L} + 2\vec{S}) \rangle$$

$$= \frac{e}{2m_e c} \langle \vec{B}_{ext} \cdot (\vec{J} + \vec{S}) \rangle$$

Of course, we should choose the axis of quantization of $\vec{J}$ to correspond with the direction of $\vec{B}_{ext}$, i.e., take

$$\vec{B}_{ext} = B_{ext}\,\hat{z}. \quad \text{Then}$$

$$\langle \vec{B}_{ext} \cdot \vec{J} \rangle = B_{ext}\,\hbar m_j .$$

What about $\langle \vec{B}_{ext} \cdot \vec{S} \rangle = B_{ext} \langle S_z \rangle$?

$$|l s j m_j\rangle = \alpha_- |l\, m_j + 1/2\rangle |\downarrow\rangle + \alpha_+ |l\, m_j - 1/2\rangle |\uparrow\rangle ,$$

$\alpha_\pm$ = Clebsch-Gordon coeff.s

i) $\underline{j = l + 1/2}$ $\quad \alpha_\pm = \sqrt{\dfrac{l \pm m_j + 1/2}{2l+1}}$

ii) $j = \ell - 1/2$

$$\alpha_\pm = \mp\sqrt{\frac{\ell \mp m_j + 1/2}{2\ell + 1}}$$

(See Lecture 9).

$$\langle n\ell j m_j | S_z | n\ell j m_j \rangle$$

$$= \frac{\hbar}{2}\left(|\alpha_+|^2 - |\alpha_-|^2\right)$$

$$= \frac{\hbar}{2}\begin{cases} \dfrac{\ell + m_j + 1/2 - (\ell - m_j - 1/2)}{2\ell+1}, & j = \ell + 1/2 \\[2ex] \dfrac{\ell - m_j + 1/2 - (\ell + m_j + 1/2)}{2\ell+1}, & j = \ell - 1/2 \end{cases}$$

$$= \frac{\pm \hbar m_j}{2\ell+1}, \quad j = \ell \pm 1/2$$

Finally, in the weak-field limit

$$E_z^{(1)} = \frac{e\hbar B_{ext}}{2m_e c}\, m_j \left(1 \pm \frac{1}{2\ell+1}\right),$$

$$j = \ell \pm 1/2$$

or $$E_z^{(1)} = g_J\, \mu_B B_{ext}\, m_j,$$

where $\mu_B \equiv \frac{e\hbar}{2m_e c} = 5.788 \times 10^{-5}\, \frac{eV}{T}$

is the <u>Bohr magneton</u> and

$$g_J = 1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}$$

is the <u>Landé g-factor</u>.

Check :

$$\pm \frac{1}{2\ell+1} = \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}, \quad j = \ell \pm 1/2$$

(yes, it's true!)

Physical model In the presence of the $\vec{L}\cdot\vec{S}$ interaction,

L_z and S_z are not conserved, but $J_z = L_z + S_z$ is. The vectors $\langle \vec{L} \rangle$ and $\langle \vec{S} \rangle$ precess about the constant vector $\langle \vec{J} \rangle$:

$$\frac{d}{dt}\langle \vec{S} \rangle = \frac{1}{i\hbar}\left\langle [\vec{S}, H_{s.o.}] \right\rangle \neq 0$$

$\frac{d}{dt}\langle \vec{L} \rangle \neq 0$ but $\frac{d}{dt}\langle \vec{J} \rangle = 0$ in an eigenstate $|j\, m_j\rangle$.

$$\frac{d}{dt}\langle \vec{S}\cdot\vec{J}\rangle = 0 \quad (\text{why?}), \text{ so}$$

we expect

$$\langle \vec{S}\rangle = \left\langle \frac{(\vec{S}\cdot\vec{J})\,\vec{J}}{\vec{J}^2} \right\rangle$$

Now $\vec{L} = \vec{J} - \vec{S}$, so

$$\vec{L}^2 = \vec{J}^2 + \vec{S}^2 - \vec{J}\cdot\vec{S} - \vec{S}\cdot\vec{J}$$

$$= \vec{J}^2 + \vec{S}^2 - 2(\vec{S}\cdot\vec{J})$$

$$\vec{S}\cdot\vec{J} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 + \vec{S}^2\right)$$

$$= \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) + 3/4 \right]$$

$$\langle \vec{S}\rangle = \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)} \langle \vec{J}\rangle$$

$$\vec{S}\cdot\vec{J} = \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) + 3/4\right]$$

$$\langle \vec{S} \rangle = \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)} \langle \vec{J} \rangle$$

$$\langle \vec{L} + 2\vec{S} \rangle = \langle \vec{J} + \vec{S} \rangle$$

$$= g_J \langle \vec{J} \rangle$$

$$\Rightarrow \quad E_Z^{(1)} = g_J \mu_B B_{ext} m_j$$

2) <u>Strong-field Zeeman effect</u>

If $B_{ext} \gg B_{int}$, the Zeeman effect dominates, and $H_{s.o.}$ can be treated as a perturbation. The "good" quantum #s are

n, ℓ, m_ℓ, and m_s (but not j and m_j because, in the presence of the external torque, the total ang. mom. is not conserved, but L_z and S_z are).

$$H_z = \frac{e}{2m_e c} B_{ext} \left(L_z + 2S_z \right).$$

The "unperturbed" energies are

$$E_{n m_\ell m_s} = -\frac{13.6\,eV}{n^2} + \mu_B B_{ext} (m_\ell + 2m_s).$$

In first-order pert. thy, the fine structure correction is

$$E^{(1)}_{f.s.} = \langle n \ell m_\ell m_s | H^{(1)}_{rel} + H^{(1)}_{s.o.} | n \ell m_\ell m_s \rangle$$

$E^{(1)}_{rel}$ is the same as before, but for $E^{(1)}_{s.o.}$, we need

$$\langle \vec{L} \cdot \vec{S} \rangle = \langle L_x \rangle \langle S_x \rangle + \langle L_y \rangle \langle S_y \rangle + \langle L_z \rangle \langle S_z \rangle$$
$$= \hbar^2 m_\ell m_s$$

$$\Rightarrow E^{(1)}_{f.s.} = \frac{13.6\,\mathrm{eV}}{n^2} \alpha^2 \left[\frac{3}{4n} - \left[\frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1/2)(\ell+1)} \right] \right]$$

The total energy is the sum of the Zeeman part and the fine-structure correction.

For intermediate fields $B_{ext} \sim B_{int}$, the perturbation is $H_Z + H_{f.s.}$, which is off-diagonal in both $|j\, m_j\rangle$ and $|m_\ell\, m_s\rangle$ bases.
$\Rightarrow$ deg. pert theory! (Griffiths 6.4.3)