Semiclassical (WKB) Approx.

In classical mechanics, one has in 1D:

\[ E = \frac{p^2}{2m} + V(x) \]

\[ p^2 = 2m \left( E - V(x) \right) \]

\[ p(x) = \pm \sqrt{2m \left( E - V(x) \right)} \]

In the semiclassical approximation, one assigns a local wavevector following de Broglie:

\[ k(x) = \frac{p(x)}{\hbar} = \pm \sqrt{\frac{2m}{\hbar^2} \left( E - V(x) \right)} \]
\[ \Psi(x) = \sqrt{g(x)} \ e^{i \int k(x) \ dx'}, \]

Let \( A(x) = \sqrt{g(x)} \), \( \Theta(x) = \int k(x) \ dx' \).

\[ \Psi(x) = A(x) \ e^{i \Theta(x)} \]

\[ \frac{d \Psi}{dx} = A'(x) \ e^{i \Theta} + i k(x) A \ e^{i \Theta} \]

\[ \frac{d^2 \Psi}{dx^2} = A''(x) \ e^{i \Theta} + 2 i k(x) A'(x) e^{i \Theta} + i k'(x) A e^{i \Theta} - k^2(x) A e^{i \Theta} \]

\[ = \Psi(x) \left[ \frac{A''}{A} + 2 i k(x) \frac{A'}{A} + i k'(x) \right] - k^2(x) \]
Schrödinger's eq. in 1D is

\[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi\]

\[\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2} (E-V(x))\psi = -k^2\psi\]

The semiclassical approx. satisfies the Sch. eq. if

\[\frac{A''}{A} = 0 \quad \text{(actually, if } \frac{A''}{A} \ll k(x)\text{)}\]

and

\[2k(x)\frac{A'}{A} + k'(x) = 0\]

\[\frac{A'}{A} = -\frac{1}{2} \frac{k'(x)}{k(x)}\]
\[ d \ln A(x) = -\frac{1}{2} \ d \ln k(x) \]

\[ \ln A(x) = \ln k(x)^{-1/2} + C' \]

\[ A(x) = \frac{e^{C'}}{\sqrt{k(x)}} = \frac{C}{\sqrt{k(x)}} \]

\[ \Rightarrow \ 
\Psi(x) = \frac{C}{\sqrt{|k(x)|}} e^{\pm i \int k(x') \ dx'} \]

where \[ k(x) = \frac{1}{\hbar} \sqrt{2m(E-V(x))} \]

The condition for validity of this so-called WKB (= Wentzel-Kramers-Brillouin) approximation is that
\[
\frac{A''}{A} \ll k^2(x) \\
\text{or } \frac{1}{\sqrt{\rho(x)}} \frac{d^2 \sqrt{\rho(x)}}{dx^2} \ll k^2(x),
\]

i.e., provided the density \( \rho(x) \) doesn't vary much in one wavelength of the de Broglie wave.

Actually, this analysis is a bit too naive, because the semiclassical approximation is an asymptotic approximation, which is not the same as dropping higher terms in a Taylor series!
The WKB method can also be used to describe the wavefunction in the classically forbidden regime.

\[ E < V(x) : \]

\[ k(x) = \pm \sqrt{2m(E-V(x))} = \pm \sqrt{2m(N(x)-E)} \]

\[ \Psi(x) = \frac{C}{\sqrt{|k(x)|}} e^{\pm \int_0^x |k(x)| \, dx'} \]

Application to tunneling:

Consider a potential barrier of
arbitrary shape but finite length in 1D:

\[
T = \frac{|J + c|}{|J_{\text{in}}|} = \frac{\hbar k m |F|^2}{\hbar k m |A|^2} = \frac{|F|^2}{|A|^2}
\]

In the tunneling region, \( V(x) > E \), the WKB wavefunction is

\[
\Psi(x) = \frac{C}{\sqrt{|V(x)|}} e^{\int_{x_{1}}^{x} \frac{|k(x)|}{dx} dx} + \frac{D}{\sqrt{|V(x)|}} e^{-\int_{x_{1}}^{x} |k(x)| dx}
\]

If the barrier is very high or wide, then the coefficient of the exponentially
growing term must be zero.

Then

$$\frac{|F|}{|A|} \sim e^{\int_{x_1}^{x_2} dx |k(x)| dx}$$

$$T = \frac{|F|^2}{|A|^2} \sim e^{2 \int_{x_1}^{x_2} dx |k(x)| dx}$$

WKB result for the tunneling probability

See application to α decay.
Example 8.2, Griffiths pp. 322-325.
General derivation

\[-\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{r}) \psi = i \hbar \frac{\partial \psi}{\partial t}\]

let \ \psi(\vec{r},t) = A(\vec{r},t) e^{\frac{i}{\hbar} R(\vec{r},t)}

where \ A, R \in \mathbb{R} \ and \ A^2 = S(\vec{r},t).

Inserting this expression for \ \psi \ into

the Schrödinger equation gives the

real and imaginary parts, gives

two equations

\[-\frac{\partial R}{\partial t} = \frac{(\nabla R)^2}{2m} + V(\vec{r}) - \frac{\hbar^2}{2m} \frac{\nabla^2 S}{\sqrt{S}}\]

\[\frac{\partial S}{\partial t} + \nabla \cdot (S \frac{\nabla R}{m}) = 0\]
The second equation is the continuity equation

\[ \frac{ds}{dt} + D \cdot \mathbf{J} = 0 \]

(Note: No approximations have been made yet!)

If we define \( \mathbf{J} = s \mathbf{V} \), then we can identify

\[ \frac{\mathbf{D} \mathbf{R}}{m} = \mathbf{V} = \text{velocity of particle} \]

The first equation, aside from the last term

\[ V_B = -\frac{e^2}{2m} \frac{D^2 \sqrt{s}}{\sqrt{s}} = \text{"quantum potential"} \]
is the Hamilton-Jacobi eq. \( \left( \right) \)
which you can find in any advanced

\textit{text} on \( \textit{classical} \) mechanics.

\( \text{(The term} \quad - \frac{\partial H}{\partial t} = E \quad \text{for} \)

\textit{time independent problems.)} \ The

3D WKB approx. results if
we neglect the quantum

total potential \( V_B(\vec{r}) \), which
gives a \textit{contribution to} \( R(\vec{r},t) \)
that is \textit{higher-order in} \( \hbar \):

\[ R(\vec{r},t) = R_0(\vec{r},t) + \hbar^2 R_2(\vec{r},t) + \ldots, \]

where \( R_0(\vec{r},t) \) is the sol'n
of 
\[ \frac{\partial R_0}{\partial t} = \frac{1}{2m} \left( \nabla R_0 \right)^2 + V(\vec{r}) \]

and

\[ -\frac{\partial R_2}{\partial t} = \frac{\partial R_0}{\partial m} \frac{\partial R_2}{\partial m} - \frac{1}{2m} \frac{D^2 \sqrt{\phi_0}}{\sqrt{\phi_0}}, \]

where \[ \frac{\partial \phi_0}{\partial t} = D_\omega \left( \frac{\partial \phi_0}{\partial m} \right), \text{ etc.} \]

Also \[ \phi = \phi_0 + \hbar^2 \phi_2 + \cdots, \text{ where} \]

\[ -\frac{\partial \phi_2}{\partial t} + D_\omega \left( \phi_0 \frac{\partial \phi_2}{\partial m} + \phi_2 \frac{\partial \phi_0}{\partial m} \right) = 0. \]

It ends up that \( \phi_2 \) is not necessarily smaller than \( \phi_0 \), but it is a rapidly oscillating function that averages to zero.