

Physics 472 Lecture 19

WKB Approximation II

Condition for validity of WKB:

$$\frac{A''(x)}{A(x)} \ll k(x)^2 = \frac{2m}{\hbar^2} (E - V(x))$$

$$A(x) = \frac{C}{\sqrt{k(x)}}$$

$$A'(x) = -\frac{1}{2} C k^{-3/2} k'(x)$$

$$A''(x) = \frac{3}{4} C k^{-5/2} (k'(x))^2 - \frac{1}{2} C k^{-3/2} k''(x)$$

$$\frac{A''}{A} = \frac{3}{4} \left(\frac{k'}{k}\right)^2 - \frac{1}{2} \frac{k''}{k}$$

$$\text{But } 2k k' = -\frac{2m}{\hbar^2} V'(x)$$

$$(k')^2 + k k'' = -\frac{m}{\hbar^2} V''(x). \quad (2)$$

Thus, WKB is valid provided the potential varies slowly:

$$\frac{\lambda}{E} \left| \frac{\partial V}{\partial x} \right| \ll 1, \quad \frac{\lambda^2}{E} \left| \frac{\partial^2 V}{\partial x^2} \right| \ll 1.$$

Turning points

Note that the WKB wavefunction is undefined at a classical turning point, where $k(x) = 0$.

WKB predicts that

$$f(x) = \frac{|C|^2}{|k(x)|} \rightarrow \infty \quad \text{at a turning point.}$$

[Does this make sense, classically?] 3

However, the potential can be linearized at a turning point, and an exact soln of the Sch. eq. can be obtained:

$$V(x) \approx E + \overbrace{\frac{\partial V}{\partial x}}^{\alpha} \Big|_{x_0} (x - x_0)$$

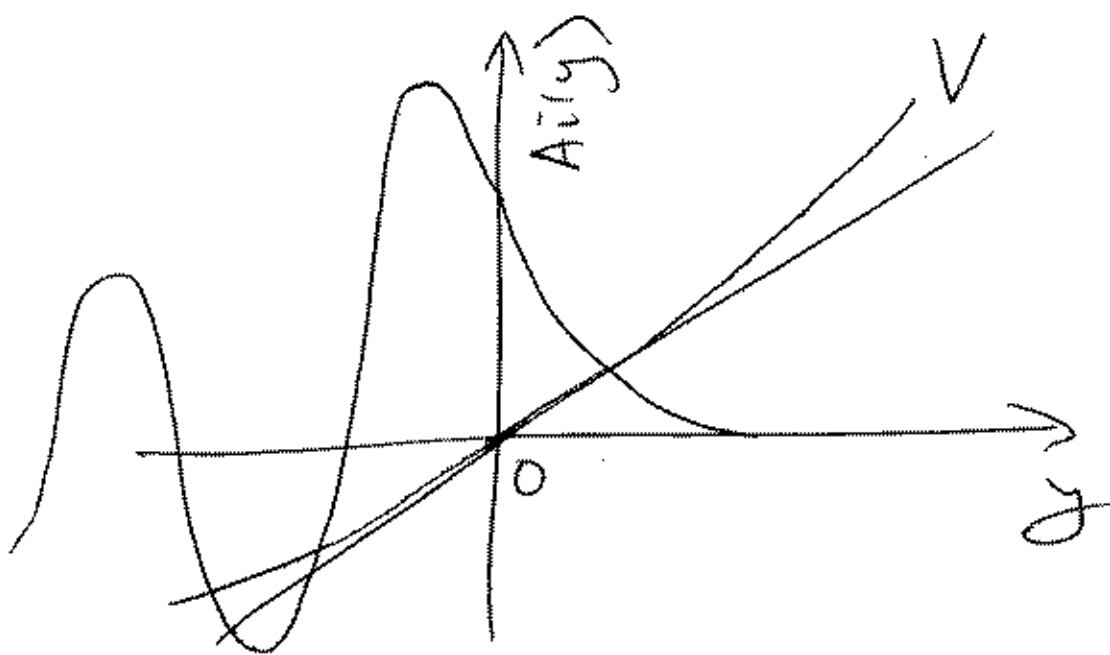
$$\Rightarrow \frac{d^2 \psi}{dx^2} - \frac{2m\alpha}{\hbar^2} (x - x_0) \psi = 0$$

$$\text{Let } y = \left(\frac{2m\alpha}{\hbar^2} \right)^{1/3} (x - x_0)$$

$$\Rightarrow \frac{d^2 \psi}{dy^2} - y \psi = 0$$

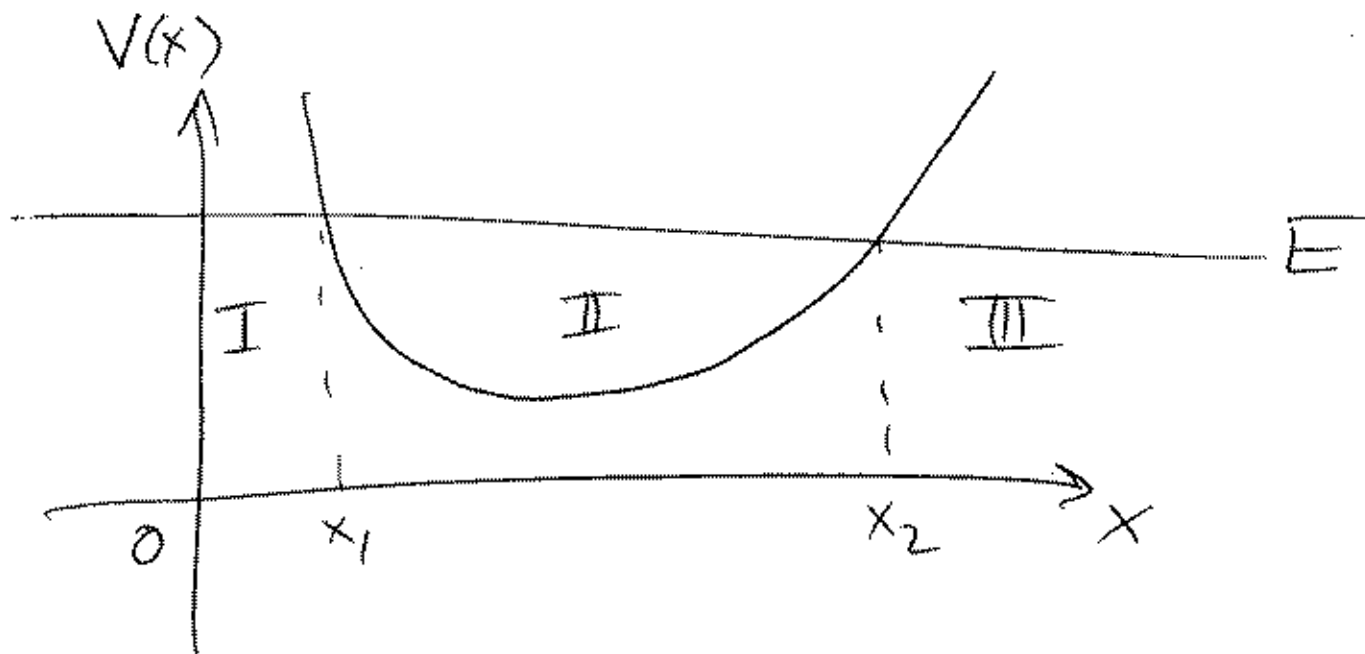
The solution of this eq. is 4
the Airy function $Ai(y)$

Connection formulas (for bound states)



For bound states, we are interested
in the Airy function that decays
to zero as $y \rightarrow \infty$, and is
oscillatory for $y < 0$, whose
asymptotic behavior is

$$Ai(y) \sim \begin{cases} \frac{1}{2\sqrt{\pi}y^{1/4}} e^{-\frac{2}{3}y^{3/2}}, & y > 0 \\ \frac{1}{\sqrt{\pi}(-y)^{1/4}} \sin\left(\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4}\right), & y < 0 \end{cases} \quad [5]$$



$$\psi_{\text{I}}(x) = \frac{C_{\text{I}}}{\sqrt{|k(x)|}} e^{\int_{x_1}^x |k(x')| dx'}$$

$$\psi_{\text{III}}(x) = \frac{C_{\text{III}}}{\sqrt{|k(x)|}} e^{-\int_{x_2}^x |k(x')| dx'}$$

In the neighborhood of the \hookleftarrow turning point $x_0 = x_2$, we have

$$p^2 = 2m(E - V) = -2m\alpha(x - x_2) \\ = -(2m\alpha\hbar)^{2/3} y$$

$$\Rightarrow -\frac{1}{\hbar} \int_{x_2}^x |P(x')| dx' = -\int_0^y \sqrt{y'} dy' \\ = -\frac{2}{3} y^{3/2}$$

Thus, the WKB wavefunction agrees with the asymptotic form of the Airy function to the right of x_2 . On the other hand, the integrand in the exponent of the oscillatory

functions in region II is 7

$$\frac{1}{h} \int_x^{x_2} P(x') dx' = - \int_0^y \sqrt{-y'} dy' = \frac{2}{3} (-y)^{3/2}$$

In order to match up at $x = x_2$,

we must take

$$\psi_{II}(x) = \frac{2C_{III}}{\sqrt{|k(x)|}} \sin\left(\int_x^{x_2} k(x') dx' + \frac{\pi}{4}\right)$$

similarly, matching solutions at

$x = x_1$ gives

$$\psi_{II}(x) = \frac{2C_I}{\sqrt{|k(x)|}} \sin\left(\int_{x_1}^x k(x') dx' + \frac{\pi}{4}\right)$$

The two expressions for ψ_{II} must be equal:

$$C_I \sin\left(\int_{x_1}^x k(x') dx' + \frac{\pi}{4}\right) = C_{III} \sin\left(\int_x^{x_2} k(x') dx' + \frac{\pi}{4}\right)$$

$$\text{But } \int_{x_1}^x k(x') dx' = \int_{x_1}^{x_2} k(x') dx' - \int_x^{x_2} k(x') dx'$$

We require

$$\int_{x_1}^{x_2} k(x') dx' = \left(n + \frac{1}{2}\right)\pi, \quad n=0, 1, 2, \dots$$

The condition that a bound state must satisfy is thus

$$\int_{x_1}^{x_2} k(x') dx' = \frac{1}{h} \int_{x_1}^{x_2} dx' \sqrt{2m(E - V(x'))} = \left(n + \frac{1}{2}\right)\pi$$

$$\text{or } \oint k(x) dx = \left(n + \frac{1}{2}\right) 2\pi \quad \text{[9]}$$

$$\oint p dx = \left(n + \frac{1}{2}\right) h$$

This is the semiclassical

(Bohr-Sommerfeld) quantization
condition.