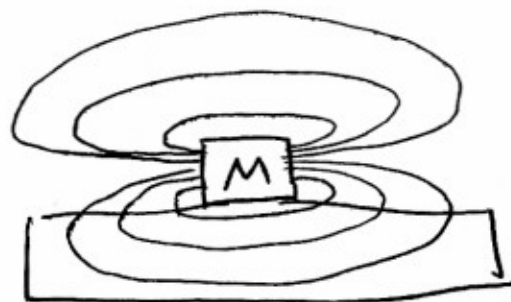


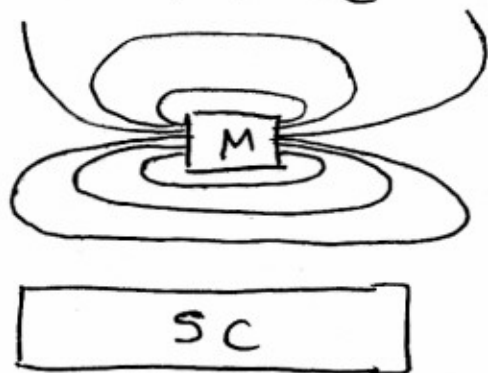
Meissner effect

$T > T_c$



magnetic field penetrates normal metal

$T < T_c$



magnetic field expelled from superconductor

Q: Can we understand the Meissner effect in terms of perfect conductivity?

Classically,
$$m \frac{d\vec{v}}{dt} = q \vec{E}$$

(charged particle of mass m in an electric field)

$$\vec{J}_e = n_s q \langle \vec{v} \rangle$$

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$$n_s = \frac{\# \text{ carriers}}{\text{volume}}$$

$q = \text{charge}$

$$n_s q \frac{d\langle \vec{v} \rangle}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d\vec{J}_e}{dt} = \frac{n_s q^2}{m} \vec{E}$$

$$\frac{d}{dt} \nabla \times \vec{J}_e = \frac{n_s q^2}{m} \nabla \times \vec{E}$$

$$\text{But } \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law})$$

$$\Rightarrow \frac{d}{dt} \left[\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} \right] = 0$$

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = \text{const.}$$

Currents flow only on the surface

of a perfect conductor, so in order 3
to force $\vec{B} = 0$ in the interior, we
must have the integration constant
in the above equation = 0.

$$\nabla \times \vec{J}_e + \frac{n_s q^2}{m c} \vec{B} = 0$$

London
equation

The London equation describes the
Meissner effect. Perfect
conductivity does not imply
the Meissner effect, since a
perfect conductor maintains a
constant field in its interior,
but this constant field
need not be zero!

The Meissner effect is a quantum effect, which occurs in superconductors, but would not occur in a perfect classical conductor. 4

In order to see why the constant in the London equation is zero, we need to see how the Schrödinger equation is modified in the presence of an (electro)magnetic field:

Maxwell's equations: (cgs units)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 4\pi \rho_e$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

Vector and scalar potentials

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$$\vec{B} = \nabla \times \vec{A}, \quad \vec{A}(\vec{r}, t) = \text{vector potential}$$

$$\vec{E} = -\nabla V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad V = \text{scalar potential}$$

Force on a charged particle

(classical):

$$\vec{F} = q \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

Gauge invariance :

$$\vec{A}' = \vec{A} + \nabla f(\vec{r}, t)$$

$$V' = V - \frac{1}{c} \frac{\partial f}{\partial t}$$

$$\vec{B}' = \vec{B}, \quad \vec{E}' = \vec{E}$$

Similar symmetry in QM

$$\psi'(\vec{r}, t) = e^{i\theta(\vec{r}, t)} \psi(\vec{r}, t)$$

$$\rho(\vec{r}, t) = |\Psi|^2 \quad \text{unchanged} \quad (6)$$

$$\text{But } \vec{p} \Psi' = e^{i\theta} (\vec{p} + \hbar \nabla \theta) \Psi.$$

(Recall $\vec{p} = \frac{\hbar}{i} \nabla =$ momentum operator.)

$$\text{Define } \vec{p}' = \frac{\hbar}{i} \nabla - \hbar \nabla \theta$$

$$\text{Then } \vec{p}' \Psi' = e^{i\theta} \vec{p} \Psi$$

$$\vec{J} = \text{Re} \left\{ \Psi^* \frac{\vec{p}}{m} \Psi \right\} = \text{Re} \left\{ (\Psi')^* \frac{\vec{p}'}{m} \Psi' \right\}$$

Physical observables (e.g. ρ , \vec{J}) are unchanged under the transformation

$$\Psi \rightarrow e^{i\theta(\vec{r}, t)} \Psi$$

$$\vec{p} \rightarrow \vec{p} - \hbar \nabla \theta$$

Schrödinger equation

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$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + gV(\vec{r}, t) \psi$$

$$\psi = e^{-i\theta} \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} + \hbar \frac{\partial \theta}{\partial t} \psi' = \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \hbar \nabla \theta \right)^2 \psi' + gV(\vec{r}, t) \psi'$$

$$i\hbar \frac{\partial \psi'}{\partial t} = \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \hbar \nabla \theta \right)^2 \psi' + g \left(V - \frac{\hbar}{g} \frac{\partial \theta}{\partial t} \right) \psi'$$

Looks like a gauge transformation with

$$f(\vec{r}, t) = \frac{\hbar c}{g} \theta(\vec{r}, t)$$

If we introduce the "kinetic momentum"

$$\vec{p}_{\text{kin}} = \frac{\hbar}{i} \nabla - \frac{g}{c} \vec{A},$$

then under a gauge transformation (8)

$$\vec{p}'_{\text{kin}} = \vec{p}_{\text{kin}} - \frac{q}{c} \nabla f = \vec{p}_{\text{kin}} - \hbar \nabla \theta \quad \checkmark$$

The gauge-invariant form of Schrödinger's equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{q}{c} \vec{A}(\vec{r}, t) \right)^2 \psi + g V(\vec{r}, t) \psi$$

Cf. Classical Hamiltonian:

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + g V,$$

$$m\vec{v} = \vec{p} - \frac{q}{c} \vec{A}$$

\vec{p} = "canonical momentum"

Quantum mechanically, the electric current is

$$\vec{J}_e = q \operatorname{Re} \left\{ \psi^* \vec{\nabla} \psi \right\}$$

$$\equiv \frac{q}{m} \operatorname{Re} \left\{ \psi^* \left(\vec{p} - \frac{q}{c} \vec{A} \right) \psi \right\}$$

$$\vec{J}_e = \frac{q}{m} \operatorname{Re} \left\{ \psi^* \left(\frac{\hbar}{i} \vec{\nabla} - \frac{q}{c} \vec{A} \right) \psi \right\}$$

In a superconductor, the charge carriers condense into a single, macroscopic wave function

$$\psi_s(\vec{r}) = \sqrt{n_s(\vec{r})} e^{i\theta(\vec{r})}$$

$$n_s(\vec{r}) = |\psi_s(\vec{r})|^2$$

$$\Rightarrow \vec{J}_e = \frac{n_s q}{m} \left(\hbar \vec{\nabla} \theta - \frac{q}{c} \vec{A} \right)$$

$$\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \nabla \times \vec{A}$$

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$$\nabla \times \vec{J}_e + \frac{n_s q^2}{mc} \vec{B} = 0 \quad !$$

London equation follows trivially from QM def. of current, provided all carriers are in the same wavefunction $\psi_s(\vec{r})$.

Q: If electrons are fermions, and must obey the Pauli exclusion principle, how can they all have the same wave function in a superconductor?!

A: They don't! The charge

Carriers in a SC are

(11)

"Cooper pairs" of electrons,

with charge $q = -2e$

and mass $m = 2m_e$. Such

pairs of electrons are bosons,

and can condense into a

single "ground state" wave

function.

Penetration depth

At the surface of a SC, the currents which screen out magnetic fields from the interior flow, and the magnetic field can penetrate a short distance.

Combining the London

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equation $\nabla \times \vec{J}_e = -\frac{n_s q^2}{mc} \vec{B}$

and Ampere's law (for static fields)

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}_e \quad \text{gives}$$

$$\nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B} = \frac{4\pi}{c} \nabla \times \vec{J}_e$$

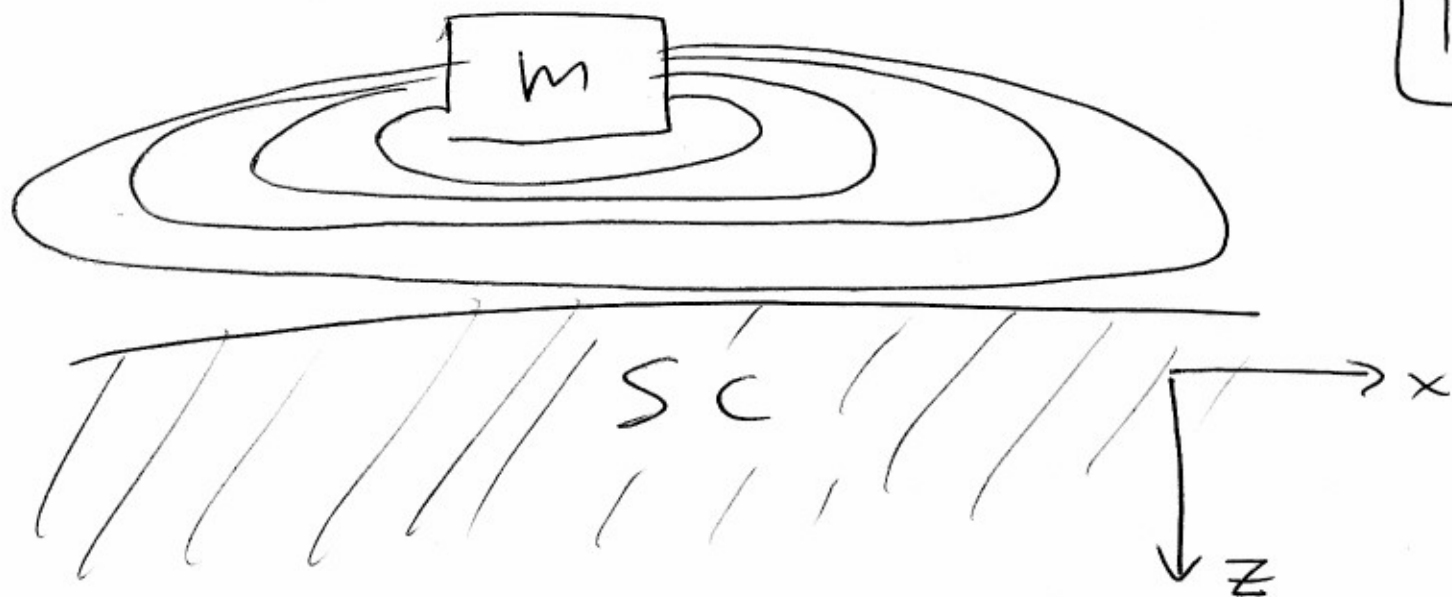
$$\Rightarrow \nabla^2 \vec{B} = \frac{4\pi n_s q^2}{mc^2} \vec{B}$$

Define the London penetration

depth

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_s q^2}}$$

$$\nabla^2 \vec{B} = \lambda_L^{-2} \vec{B}$$



Neglecting edge effects,

$$\vec{B}(r) = \hat{x} B_x(z)$$

$$\frac{d^2 B_x}{dz^2} = \frac{1}{\lambda_L^2} B_x(z)$$

Sol'n: $B_x(z) = B_x(0) e^{-z/\lambda_L}$

⇒ Field penetrates exponentially,
with a decay length λ_L .