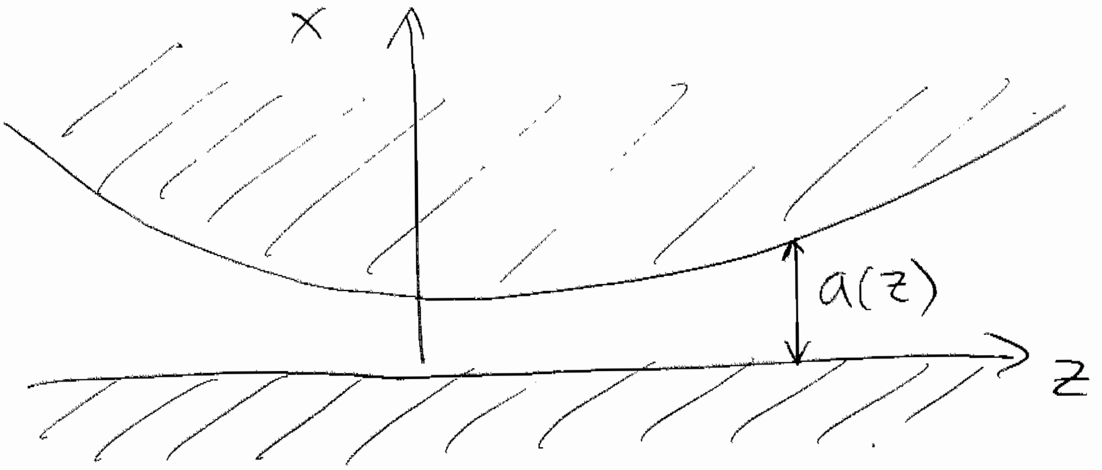
# Physics 472 Lecture 23

## Applications of the adiabatic approx.

### 1) Motion on a strip of varying width




$$E\,\Psi(x,z) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\Psi(x,z)$$

B.C.s $\qquad \Psi(0,z) = 0$

$$\Psi(a(z), z) = 0$$

Adiabatic ansatz

$$\Psi(x,z) = \psi(z)\,\chi_z(x)$$

$$-\frac{2mE}{\hbar^2}\psi\chi_z = \frac{d^2\psi}{dz^2}\chi_z(x) + \psi(z)\frac{\partial^2\chi_z}{\partial x^2}$$

assumptions: $\frac{\partial\chi_z}{\partial z} \ll \frac{\partial\chi_z}{\partial x}$

and $\psi\frac{\partial\chi_z}{\partial z} \ll \frac{\partial\psi}{\partial z}\chi_z(x)$

$$-\frac{2mE}{\hbar^2} = \frac{\psi''(z)}{\psi(z)} + \frac{\frac{\partial^2\chi_z(x)}{\partial x^2}}{\chi_z(x)}$$

$$\frac{1}{\chi_z}\frac{\partial^2\chi_z(x)}{\partial x^2} = \text{const.} = -K^2$$

solution: $\chi_z(x) = A\sin(Kx) + B\cos(Kx)$

B.C.s: $0 = \chi_z(0) = B$

$0 = \chi_z(a(z)) = A\sin(Ka(z))$

$\Longrightarrow \quad Ka(z) = n\pi, \quad n = 1, 2, 3, \ldots$

$$\boxed{K_n = \frac{n\pi}{a(z)}, \quad \chi_z^{(n)}(x) = \sqrt{\frac{2}{a(z)}}\sin\left(\frac{n\pi x}{a(z)}\right)}$$



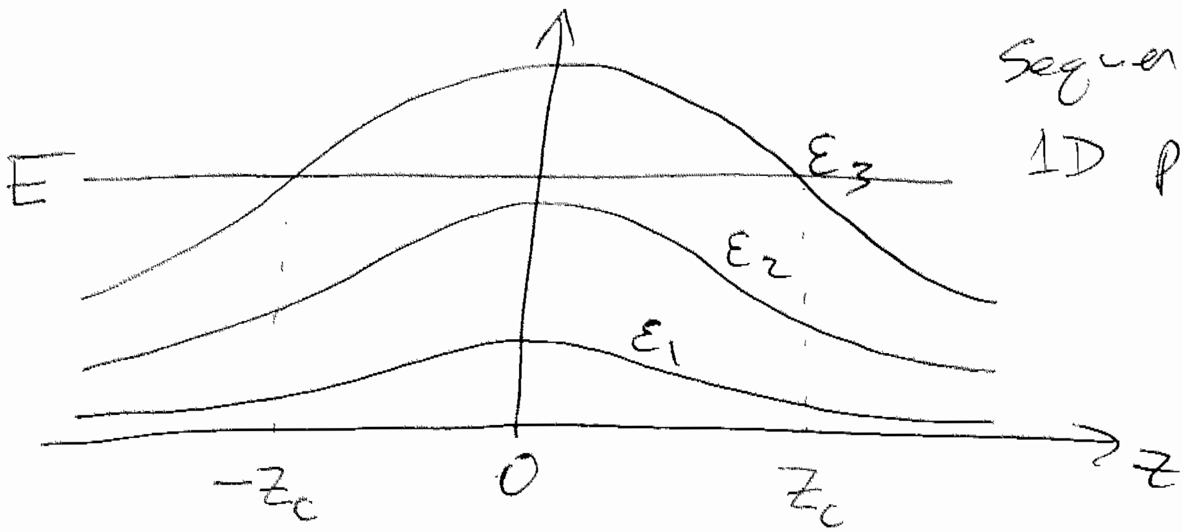
$$\boxed{-\frac{\hbar^2}{2m}\frac{1}{\chi_z}\frac{\partial^2 \chi_z(x)}{\partial x^2} = \frac{\hbar^2\pi^2 n^2}{2m\, a(z)^2} \equiv \mathcal{E}_n(z)}$$

Equation for $\psi(z)$:

$$\frac{\psi''(z)}{\psi(z)} = -\frac{2m}{\hbar^2}\left(E - \mathcal{E}_n(z)\right)$$

Equiv. to 1D Sch. eq. w/ $V(z) = \mathcal{E}_n(z)$

$$E\,\psi(z) = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dz^2} + \mathcal{E}_n(z)\,\psi(z)$$




Sequence of 1D problems

Check:

$$\frac{\partial \chi_z}{\partial x} = \frac{n\pi}{a}\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right)$$

$$\frac{\partial \chi_z}{\partial z} = -\frac{n\pi x}{a^2}\,a'(z)\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) - \frac{1}{2}\sqrt{\frac{2}{a}}\,\frac{a'(z)}{a}\sin\left(\frac{n\pi x}{a}\right)$$

$$\frac{\partial \chi_z}{\partial z} = -\frac{a'(z)}{a(z)}\left[x\,\frac{\partial \chi_z}{\partial x} + \frac{1}{2}\chi_z\right]$$

Condition for validity

$$\lambda\,\frac{a'(z)}{a(z)} \ll 1 \qquad (\lambda = \text{de Broglie wavelength})$$

i.e., $\left|\dfrac{da(z)}{dz}\right| \ll \dfrac{a(z)}{\lambda}$

Quantum # $n$ is conserved

(no transitions $n \rightarrow n'$)

But if $|a'(z)|$ is small, we can also use the WKB approx. to solve for $\psi(z)$:

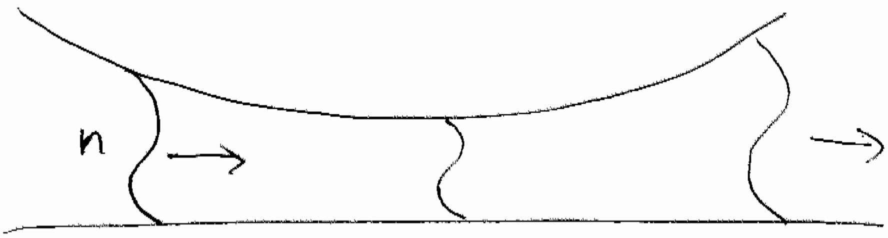
$$k_n(z) = \pm\sqrt{\frac{2m}{\hbar^2}(E - \mathcal{E}_n(z))}$$

$$\psi_n^{(\pm)}(z) = \frac{1}{\sqrt{|k_n(z)|}}\, e^{\pm i \int^z k_n(z')\,dz'}$$

General solution:

$$\Psi(x, z) = \sum_{n=1}^{\infty} \chi_z^{(n)}(x)\left[C_n^{(+)}\psi_n^{(+)}(z) + C_n^{(-)}\psi_n^{(-)}(z\right.$$

<u>Scattering</u>:



Incident wave:

$$\Psi_{in}(x,z) = A\,\chi_z^{(n)}(x)\,\frac{e^{i\int_0^z k_n(z')dz'}}{\sqrt{|k_n(z)|}}$$

$$j_{in}(z) = \int_0^{a(z)} dz\ \mathrm{Re}\left\{\Psi_{in}^*\,\frac{\hbar}{im}\,\frac{\partial \Psi_{in}}{\partial z}\right\}$$

$$\simeq |A|^2\,\frac{\hbar}{m}$$

Transmitted wave:

$$\Psi_{tr}(x,z) = \sum_{n'} B_{n'}\,\chi_z^{(n')}(x)\,\frac{e^{i\int_0^z k_{n'}(z')dz'}}{\sqrt{|k_{n'}(z)|}}$$

$n$ conserved $\rightarrow$ $B_{n'} = B\,\delta_{nn'}$

$$j_{tr} = |B|^2\,\frac{\hbar}{m}$$

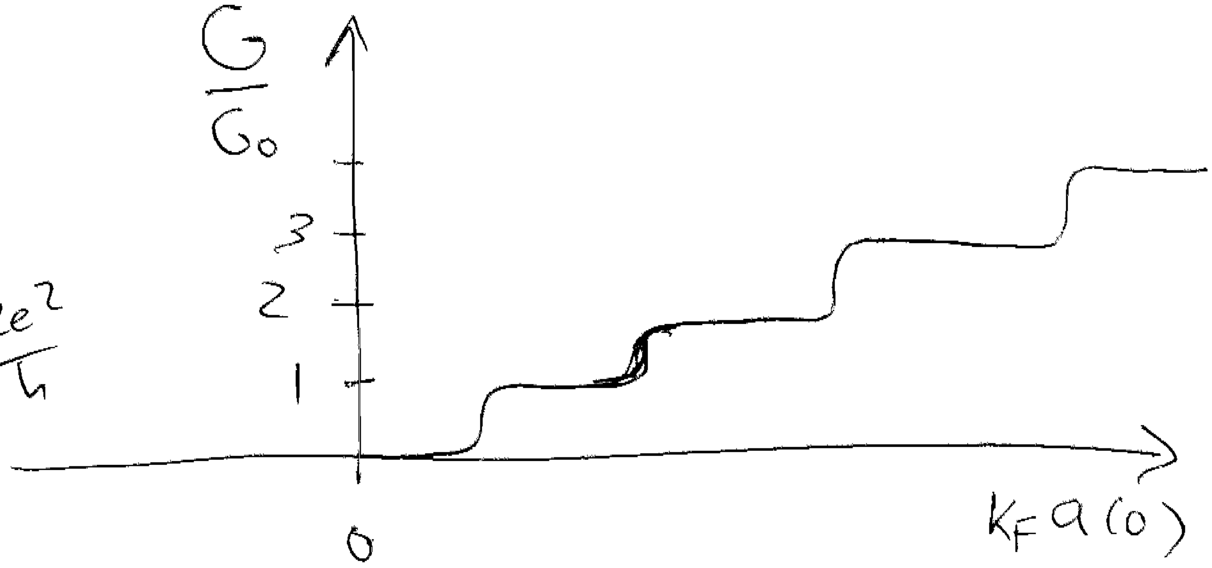
Transmission probability

$$T_{nn'} = \frac{|B_{n'}|^2}{|A|^2} = \frac{|B|^2}{|A|^2}\delta_{nn'}$$

$$\frac{|B|^2}{|A|^2} \approx e^{-2\int_{-z_c}^{z_c} |k_n(z')|\,dz'}$$

## Electrical conductance

$$\frac{1}{R} \equiv G = \frac{2e^2}{h}\sum_{nn'} T_{nn'}(\varepsilon_F)$$




$G_0 = \frac{2e^2}{h}$

## 2) Berry's phase

$$\Psi(t) = e^{i[\theta_n(t) + \gamma_n(t)]} \psi_n(t),$$

where $\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')\,dt'$

is the dynamic phase, and

$$\gamma_n(t) = i\int_0^t \left\langle \psi_n(t') \middle| \frac{\partial}{\partial t'} \psi_n(t') \right\rangle dt'$$

is the geometric phase.

Berry's phase is the geometric phase accumulated when the Hamiltonian is slowly varied until it returns to its initial form.

Suppose a single parameter $R(t)$ is varied. Then

$$\frac{\partial \psi_n(t)}{\partial t} = \frac{\partial \psi_n}{\partial R} \frac{dR}{dt}$$

and

$$\gamma_n(t) = i \int_0^t \langle \psi_n | \frac{\partial \psi_n}{\partial R} \rangle \frac{dR}{dt'} dt'$$

$$= i \int_{R_i}^{R_f} \langle \psi_n | \frac{\partial \psi_n}{\partial R} \rangle dR$$

Berry's phase

$$\gamma_n = i \oint \langle \psi_n | \frac{\partial \psi_n}{\partial R} \rangle dR = 0$$

in that case.

In general, however, the

Hamiltonian may depend on a number of parameters

$R_1(t), R_2(t), \ldots, R_N(t)$,

in which case

$$\frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial \psi_n}{\partial R_2}\frac{dR_2}{dt} + \cdots + \frac{\partial \psi_n}{\partial R_N}\frac{dR_N}{dt}$$

$$= (\nabla_R \psi_n) \cdot \frac{d\vec{R}}{dt}.$$

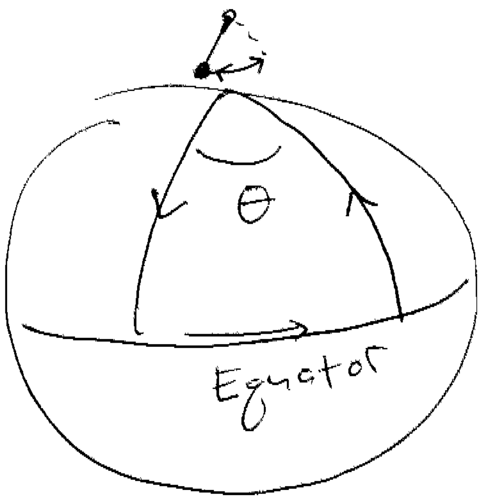
Then

$$\gamma_n(t) = i \int_{\vec{R}_i}^{\vec{R}_f} \langle \psi_n | \nabla_R \psi_n \rangle \cdot d\vec{R}$$

and Berry's phase is

$$\gamma_n = i \oint \langle \psi_n | \nabla_R \psi_n \rangle \cdot d\vec{R}$$

# Example: Foucault's pendulum

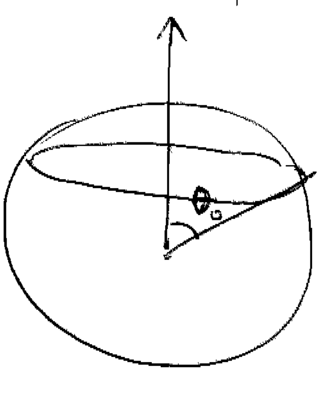



Plane of oscillation would rotate by an angle $\theta$ if the pendulum were carried adiabatically around the path shown.

Area enclosed by path

$$A = \frac{1}{2}\frac{\theta}{2\pi}4\pi R^2 = \theta R^2$$

$$\Omega = \frac{A}{R^2} = \theta \quad \text{(solid angle)}$$




$$\Omega = \int \sin\theta \, d\theta \, d\phi$$

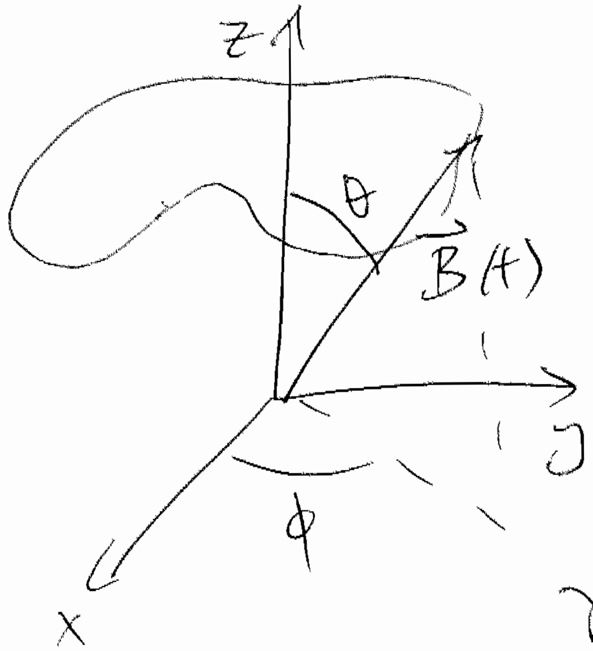
$$= 2\pi(-\cos\theta)\Big|_0^{\theta_0}$$

$$= 2\pi(1-\cos\theta_0)$$

Quantum mechanically, this would give rise to a Berry phase

$$\gamma = \Omega.$$

A typical QM example is that of an electron in a magnetic field whose direction varies slowly with time:




$$\chi_+ = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

If $\vec{B}(t)$ is taken around a closed loop,

$$\gamma_+ = -\Omega/2, \quad \gamma_- = +\Omega/2$$

$$\gamma_- - \gamma_+ = \Omega$$ (Just like Foucault's pendulum!)