Scattering in 3D
(a brief introduction)

Three-dimensional scattering is treated in depth in Chapter 11 of Griffiths. To get a brief overview of the subject, let's consider scattering as an application of Fermi's golden rule:

Wave packet

\[ \rightarrow \hat{K} \]

Mapping

\[ \frac{\vec{k}'}{V} \]

Large box of volume \( V \)
What is the rate at which the potential $U(r)$ causes transitions from the initial plane wave of wave vector $\mathbf{k}$, $\psi_k^i(r) = \sqrt{\frac{\lambda}{V}} e^{i\mathbf{k}\cdot\mathbf{r}}$, to a scattered wave $\psi_k^f(r) = \sqrt{\frac{\lambda}{V}} e^{i\mathbf{k}\cdot\mathbf{r}}$?

$$\langle k' | U | k \rangle = \int \frac{d^3r}{\sqrt{V}} e^{-i\mathbf{k}\cdot\mathbf{r}} U(r) e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= \frac{1}{\sqrt{V}} \tilde{U}(k' - k)$$

$$\Gamma_{k \rightarrow k'} = \frac{2\pi}{\hbar} |\langle k' | U | k \rangle|^2 \delta(E_{k'} - E_k)$$

Fermi's golden rule
Suppose we ask for the rate of scattering $dP$ into a small solid angle $d\Omega$:

$$dP = \sum_{\vec{k} \to \vec{k'}} \Gamma_{\vec{k} - \vec{k'}}$$

There are

$$V \frac{d^3k'}{(2\pi)^3} = V \frac{m|\vec{k'}|}{(2\pi)^3 h^2} \, d\Omega' \, d\varepsilon_{k'}$$

states in the volume $d^3k'$ of phase space, and therefore

$$V \frac{m|k'|}{(2\pi)^3 h^2} \, \text{states per unit energy per unit solid angle}.$$

$$\sum_{\vec{k} \to \vec{k'}} \rightarrow \int_{0}^{\infty} \frac{V m k'}{(2\pi)^3 h^2} \, d\varepsilon_{k'}$$
\[ d\Gamma = \frac{2\pi}{\hbar} \frac{|\tilde{\mathcal{U}}(\vec{k'}-\vec{k})|^2}{\sqrt{2}} \frac{\sqrt{m_k}}{(4\pi \hbar^2)^{3/2}} \ dv \]

\[ d\Gamma = \frac{d\Omega'}{V} \frac{mk}{4\pi^2 \hbar^2} |\tilde{\mathcal{U}}(\vec{k'}-\vec{k})|^2 \]

The incident flux of particles in state \( \vec{k} \) is

\[ \tilde{J}_{in} = \frac{\hbar \vec{k}}{m} \frac{1}{V} \]

\[ \frac{d\sigma}{d\Omega} = \frac{d\Gamma}{|\tilde{J}_{in}| \ d\Omega} = \frac{m^2}{4\pi^2 \hbar^4} |\tilde{\mathcal{U}}(\vec{k'}-\vec{k})|^2 \]

Differential scattering cross section
This result is referred to as the Born approximation for the differential scattering cross section.

See Griffiths, Ch. 11 for examples.