Physics 472 Lecture 3

Electron in a uniform static magnetic field

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + qV$$

For static fields, a convenient gauge is the Coulomb gauge

$$\nabla \cdot \vec{A} = 0$$

Recall $\quad [f(x), p_x] = i\hbar \frac{df}{dx}$

$$\Rightarrow \quad \vec{A}\cdot\vec{p} - \vec{p}\cdot\vec{A} = i\hbar \nabla\cdot\vec{A} = 0$$

The vector potential commutes with momentum in the Coulomb gauge.

$$H = \frac{\vec{p}^2}{2m} - \frac{q}{mc}\vec{A}\cdot\vec{p} + \frac{q^2}{2mc^2}\vec{A}^2 + qV$$

For a constant field $\vec{B}$, one can furthermore choose

$$\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B} \qquad \text{(symmetric gauge)}$$

$\Rightarrow$ Verify $\nabla\times\vec{A} = \vec{B}$

$$\vec{A}\cdot\vec{p} = -\frac{1}{2}(\vec{r}\times\vec{B})\cdot\vec{p} = \frac{1}{2}\vec{B}\cdot(\vec{r}\times\vec{p})$$

$$= \frac{1}{2}\vec{B}\cdot\vec{L} \qquad \left(\text{we used that } [\vec{B},\vec{r}]=0\right)$$

Furthermore,

$$\vec{A}^2 = \frac{1}{4}(\vec{r}\times\vec{B})^2 = \frac{1}{4}\left[r^2B^2 - (\vec{r}\cdot\vec{B})^2\right]$$

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{q}{2mc}\vec{B}\cdot\vec{L} + \frac{q^2}{8mc^2}\left[r^2B^2 - (\vec{r}\cdot\vec{B})^2\right] + qV$$

For sufficiently small fields, the term linear in $\vec{B}$ will dominate if $\langle \vec{r}^{\,2} \rangle$ is bounded. Note that

$$\vec{\mu} = \frac{q}{2mc}\vec{L} \quad = \text{magnetic moment}$$

$$\Delta H = -\vec{\mu}\cdot\vec{B} + \mathcal{O}(B^2)$$

$\Rightarrow$ Normal Zeeman effect

Let's take $\vec{B} = B\hat{z}$.

$$H = \frac{\vec{p}^2}{2m} - \frac{qB}{2mc} L_z + \frac{q^2B^2}{8mc^2}(x^2+y^2) + qV(\vec{r})$$

The <u>Zeeman splitting</u> is

$$\Delta E = \frac{q\hbar B}{2mc} = \hbar\omega_L$$

$$\omega_L = \frac{eB}{2mc} = \text{Larmor frequency}$$

<u>Relative size of linear and quadratic terms</u>

For atomic systems $\langle x^2+y^2\rangle \sim a_0^2$

$$\frac{\frac{e^2B^2}{8mc^2}\langle x^2+y^2\rangle}{\Delta E_{\text{Zeeman}}} \sim \frac{e^2B^2a_0^2/8mc^2}{e\hbar B/2mc}$$

$$= \frac{eBa_0^2}{4\hbar c}$$

$$= \frac{eB}{4\hbar c}\left(\frac{\hbar^2}{me^2}\right)^2 = \frac{\hbar^3 B}{4m^2e^3c}$$

$$= \left(\frac{\hbar c}{e^2}\right)^2 \frac{\Delta E_{\text{Zeeman}}}{2mc^2} \sim \frac{B}{9\times 10^9\,\text{gauss}}$$

(Note: $\frac{e^2}{\hbar c} \sim \frac{1}{137}$ fine structure constant)

Landau levels

While the quadratic term in $\vec{B}$ can be neglected for bounded systems, such as atoms, in weak fields, this is not true for free particles.

Consider a two-dimensional electron gas in the x-y plane in a field $\vec{B} = B\,\hat{z}$.

Instead of the symmetric gauge, it is convenient, for this problem to work in the Landau gauge

$$\vec{A} = (-yB, 0, 0)$$

$$H = \frac{1}{2m}\left(p_x - \frac{eBy}{c}\right)^2 + \frac{p_y^2}{2m}$$

$[p_x, H] = 0$, so seek sol'n of the form $\psi(x,y) = e^{ikx}\chi(y)$.

$$H\psi(x,y) = \frac{1}{2m}\left(\hbar k - \frac{eBy}{c}\right)^2\psi + \frac{p_y^2}{2m}\psi$$

$$E\chi(y) = \left[\frac{p_y^2}{2m} + \frac{1}{2m}\left(\hbar k - \frac{eBy}{c}\right)^2\right]\chi$$

$$E\chi(y) = \frac{p_y^2}{2m}\chi + \frac{eB^2}{2mc^2}\left(y - \frac{\hbar ck}{eB}\right)^2\chi$$

$$E\chi = \left[\frac{p_y^2}{2m} + \frac{m\Omega^2}{2}(y-y_0)^2\right]\chi$$

$$\boxed{\Omega = \frac{eB}{mc} = \text{cyclotron frequency}}$$

$$y_0 = \frac{\hbar c k}{eB}$$

$\Rightarrow$ shifted harmonic oscillator

$$\boxed{E_n = \hbar \Omega \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \infty}$$

Landau levels

Degeneracy

suppose systems has dimensions L_x, L_y. Impose periodic BCs in x-direction.

$$e^{ikL_x} = 1 \Rightarrow k = \frac{2\pi n_x}{L_x}$$

Moreover, $0 \leq y_0 \leq L_y$

$$0 \leq \frac{\hbar c}{eB} \frac{2\pi n_x}{L_x} \leq L_y$$

$$0 \leq n_x \leq \frac{eB}{hc} L_x L_y$$

Total number of states in a given Landau level of energy $E_n = \hbar\Omega(n + 1/2)$ is thus

$$N_n = \frac{eB}{hc} A$$

$$N_n = \frac{BA}{\phi_0}, \quad \phi_0 = \frac{hc}{e}$$

(normal flux quantum)

Filling factor ν

For a total # N of electrons, the total # ν of Landau levels populated is

$$\nu = \frac{N}{N_n} = \frac{N\phi_0}{BA}$$

$$= \frac{\text{\# electrons}}{\text{flux quantum}}$$