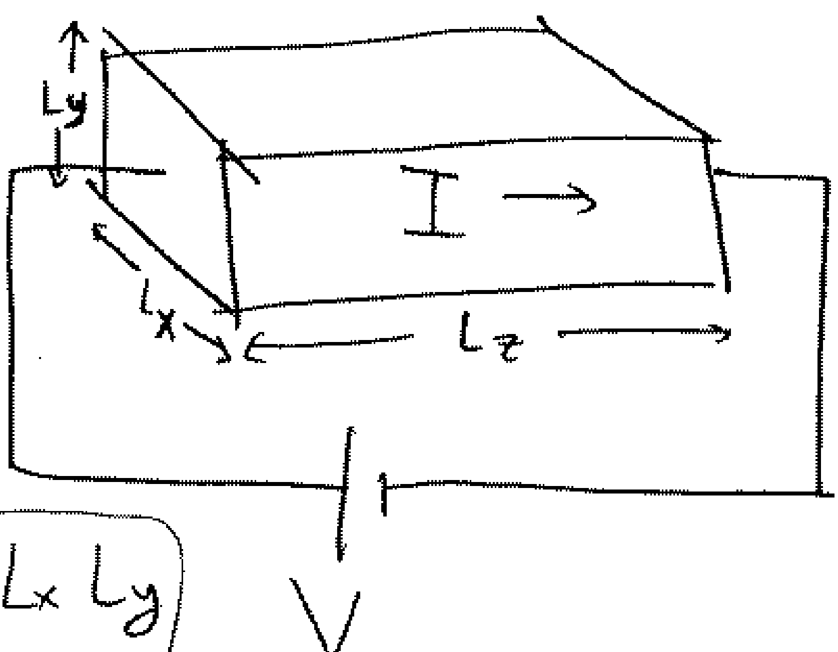
Physics 472 Lecture 4

Quantum transport

For a conductor with characteristic dimensions $L_x, L_y, L_z \gg \ell, \lambda_F$ where ℓ is the electron mean free path, transport can be described by the Boltzmann equation. The electrical resistance is given by Ohm's law

$$R = \rho \frac{L_z}{A}$$



$$V = IR$$

$$A = L_x L_y$$

$L_i \gg \lambda_F$ = Fermi wavelength

is necessary because the Boltzmann equation neglects wave mechanical effects.

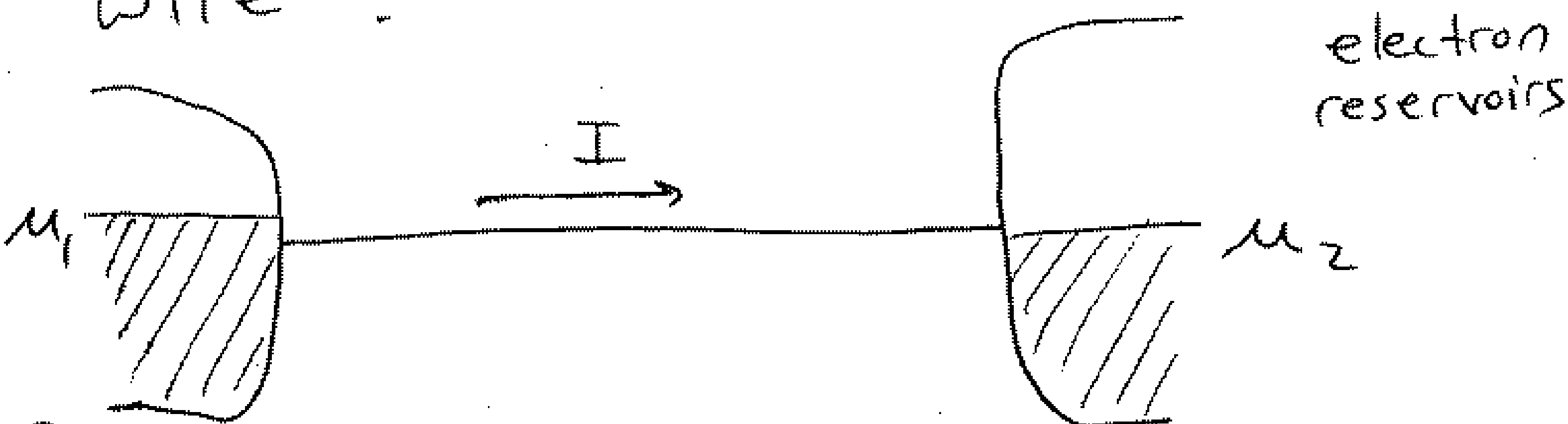
The opposite limit

$L_i \ll \ell$, $L_i \sim \lambda_F$ is

the extreme quantum limit of <u>ballistic transport</u>.

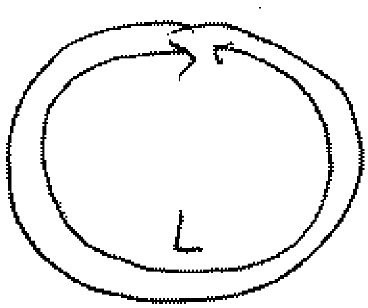
It turns out that transport can also be computed straightforwardly in this limit, but using the Schrödinger equation.

Q: What is the resistance of a perfect one-dimensional wire?



$$I = e v_F \frac{\partial n}{\partial E} (\mu_1 - \mu_2)$$

$\frac{\partial n}{\partial E}$ = density of states per unit length (unidirectional))



$$\psi(x) = A e^{ikx}$$

$$\psi(x+L) = \psi(x) \Rightarrow e^{ikL} = 1$$

$$k = \frac{2\pi N}{L}, \quad N \in \mathbb{Z}$$

$$\frac{\partial n}{\partial E} = \frac{\partial (N/L)}{\partial k} \frac{1}{\frac{\partial E}{\partial k}}$$

$$= \frac{1}{2\pi} \frac{1}{\hbar v_F} = \frac{1}{h v_F}$$

$$I = \frac{e}{h}(\mu_1 - \mu_2) = \frac{e^2}{h} V$$

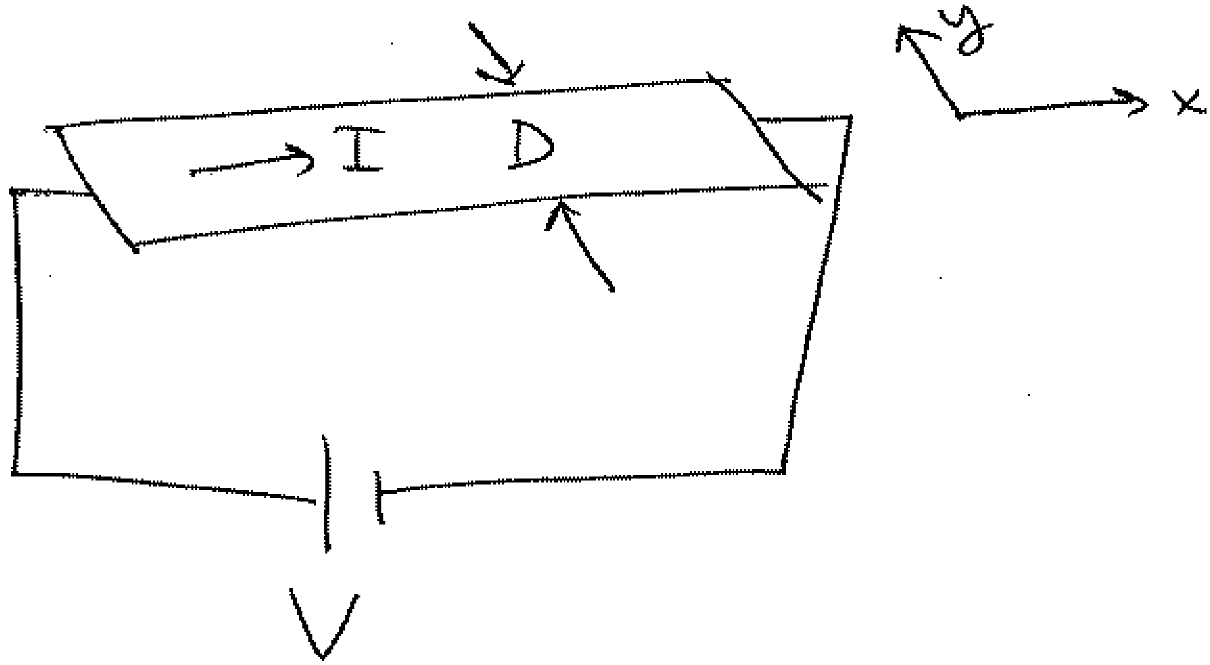
$$V = IR \qquad \boxed{R = \frac{h}{e^2}}$$

$$\boxed{\frac{h}{e^2} = 25{,}812.8\ \Omega}$$

sometimes this unit of resistance is known as the "von Klitzing," after the Nobel laureate who first measured it accurately in ca. 1980.

Contrary to the expectation of ohm's law, R is independent of the length of the wire!

Q: What is the resistance of a perfect two-dimensional wire of width D?



Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi(x,y)=E\,\Psi(x,y)$$

Boundary conditions:

$$\Psi(x,0)=\Psi(x,D)=0$$

Separation of variables:

$$\Psi(x,y)=\psi(x)\,\phi(y)$$

$$\frac{1}{\psi}\frac{d^2\psi}{dx^2}+\frac{1}{\phi}\frac{d^2\phi}{dy^2}=\frac{-2mE}{\hbar^2}$$

$$\psi(x)=e^{ikx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dy^2}=\left(E-\frac{\hbar^2k^2}{2m}\right)\phi$$

Solution:

$$\phi(y) = A\sin(py)$$

$$p^2 = \frac{2mE}{\hbar^2} - k^2$$

B.C.: $\sin pD = 0$

$pD = n\pi \qquad p = \frac{n\pi}{D}$

$$\boxed{\begin{aligned} E &= \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2 \pi^2 n^2}{2mD^2} \\ E &= \frac{\hbar^2 k^2}{2m} + \varepsilon_n \end{aligned}}$$

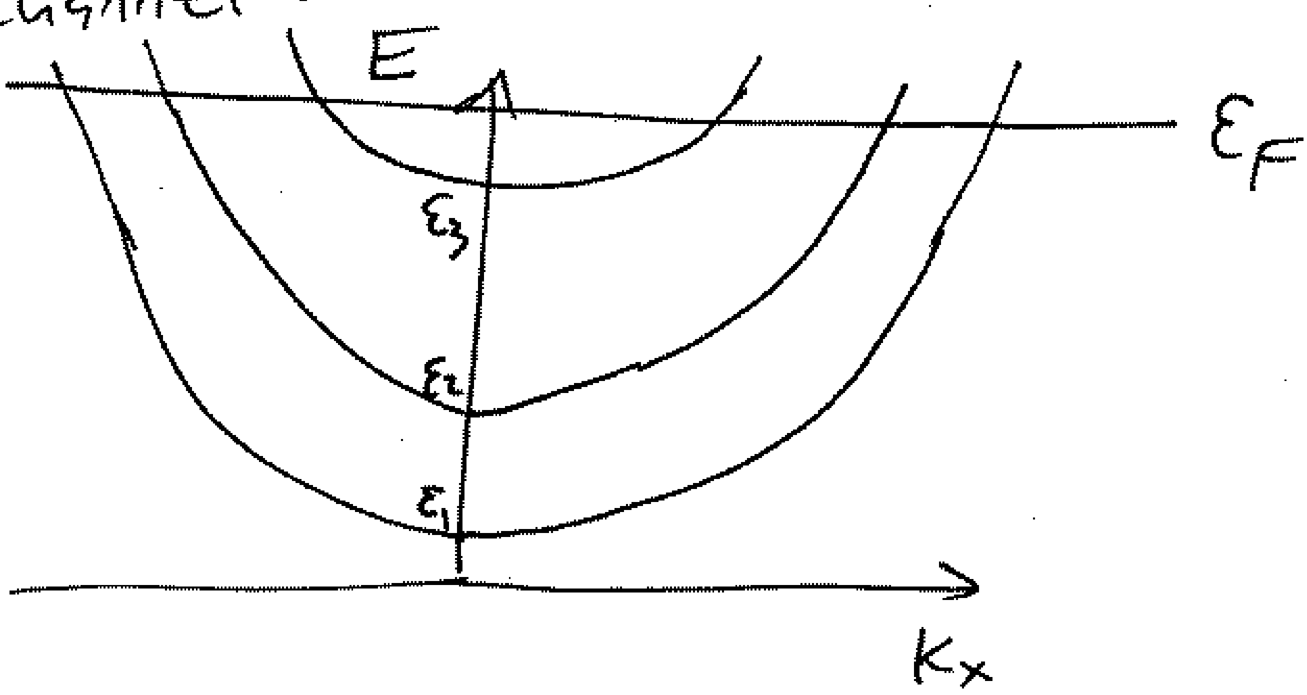
How many transverse states are there with $\varepsilon_n < \varepsilon_F$?

$$\frac{\hbar^2 \pi^2 N^2}{2m D^2} \leq \frac{\hbar^2 k_F^2}{2m}$$

$$n \leq \frac{k_F D}{\pi}$$

$$N = \max\{n\} = \mathrm{Int}\left\{\frac{k_F D}{\pi}\right\}$$

# of modes grows roughly proportional to D. Each mode n acts as a one-dimensional channel:



$$I = N\frac{e^2}{h}V$$

$$R = \frac{h}{Ne^2} \qquad G = R^{-1} = N\frac{e^2}{h}$$

Landauer formula

In general, if the quantum mechanical probability for an electron in channel n to traverse the wire is T_n, the conductance is

$$G = \frac{e^2}{h}\sum_n T_n .$$

Conductance Quantization

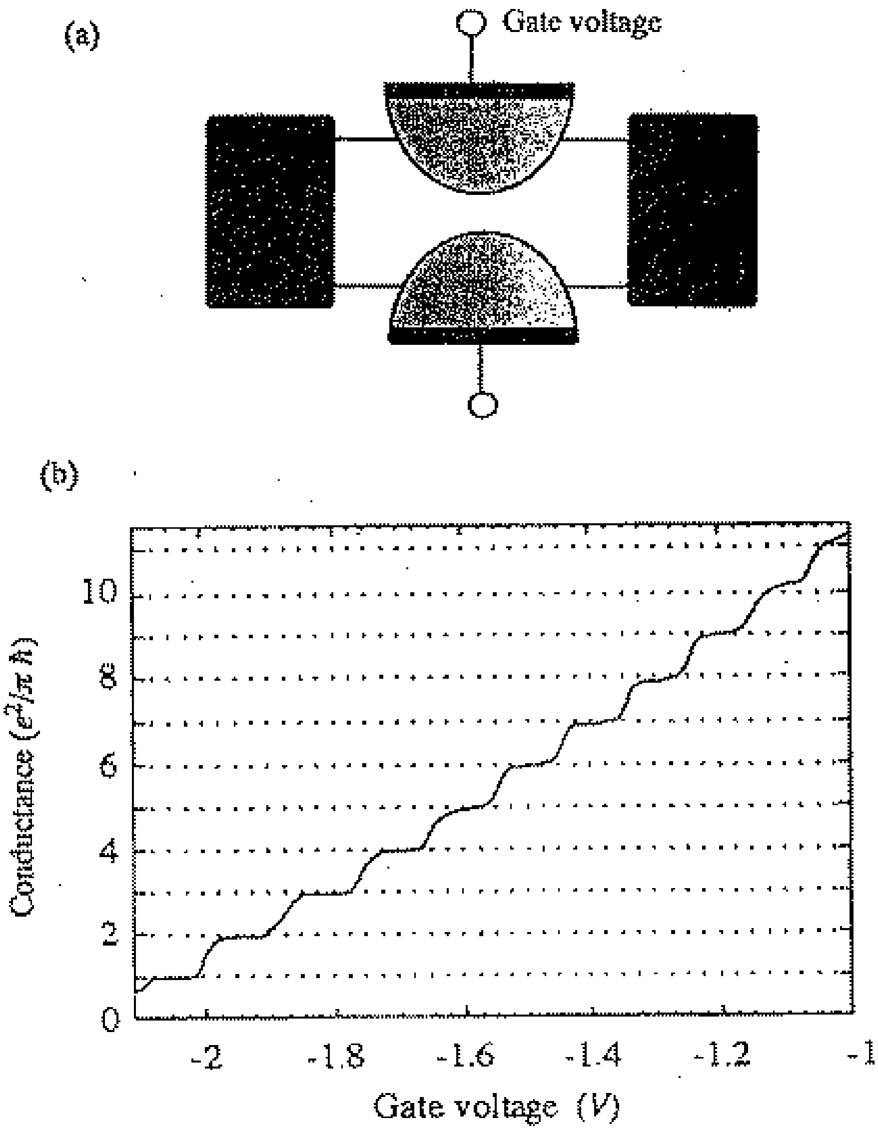


Fig. 2.1.2. Quantized conductance of a ballistic waveguide. (a) A negative voltage on a pair of metallic gates (called the split-gate configuration) is used to deplete and narrow down the constriction progressively. (b) Measured conductance vs. gate voltage. The measured resistance also includes a series resistance due to the wide regions connecting the constriction to the contacts. This series resistance is measured separately by removing the negative voltage on the gates and is subtracted off before plotting. Reproduced with permission from B. J. van Wees *et al.* (1988), *Phys. Rev. Lett.*, **60**, 848. Similar results were reported simultaneously by D. Wharam *et al.* (1988). *J. Phys. C*, **21**, L209.

Q: What is the resistance of a perfect 3-D wire with a square cross section $D^2 = A$?

$$E = \frac{\hbar k^2}{2m} + \frac{\hbar^2 \pi^2}{2mA}(n^2+m^2)$$

# of modes with $\varepsilon_{nm} < \varepsilon_F$?

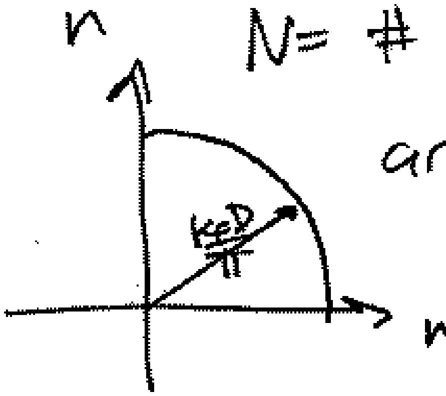
$$\varepsilon_{nm} = \frac{\hbar^2 \pi^2}{2mA}(n^2+m^2)$$

$$\frac{\hbar^2 \pi^2}{2mA}(n^2+m^2) < \frac{\hbar^2 k_F^2}{2m}$$

$$n^2+m^2 < \frac{k_F^2 A}{\pi^2}$$

$$\sqrt{n^2+m^2} < \frac{k_F D}{\pi}$$

$N = \#$ of modes



$$\text{area} = \frac{\pi}{4} r^2$$

$$= \frac{\pi}{4}\left(\frac{k_F D}{\pi}\right)^2 = \frac{k_F^2 A}{4\pi}$$

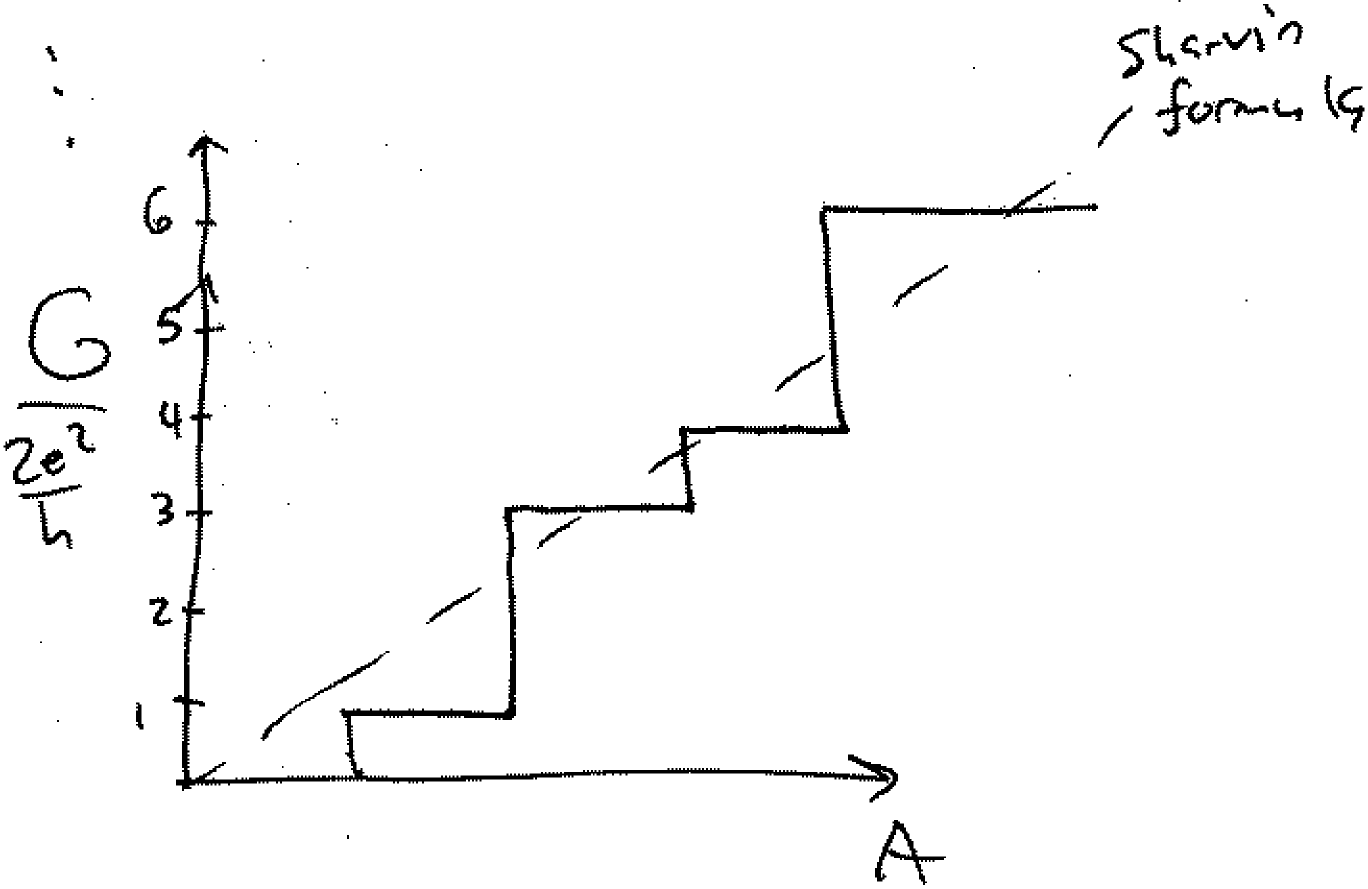
$$G = N\frac{e^2}{h} \approx \frac{k_F^2 A}{4\pi}\frac{e^2}{h}$$

If $\vec{B} = 0$, and charge carriers have spin-1/2, we must multiply G by 2:

$$G \simeq \frac{2e^2}{h}\frac{k_F^2 A}{4\pi} \quad \text{(Sharvin formula)}$$

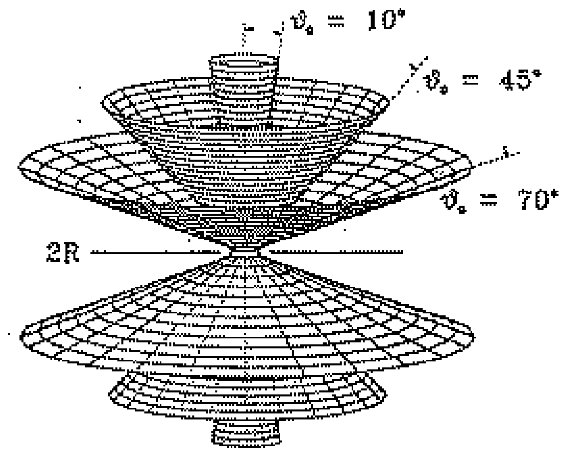
Like Ohm's law, $G \propto A$; unlike Ohm's law, G indep. of L!

mode		$\varepsilon \frac{2mA}{\pi^2\hbar^2}$
1	1	2
1	2	5
2	1	5
2	2	8
1	3	10
3	1	10
⋮		



Calculation for hyperbolic constriction

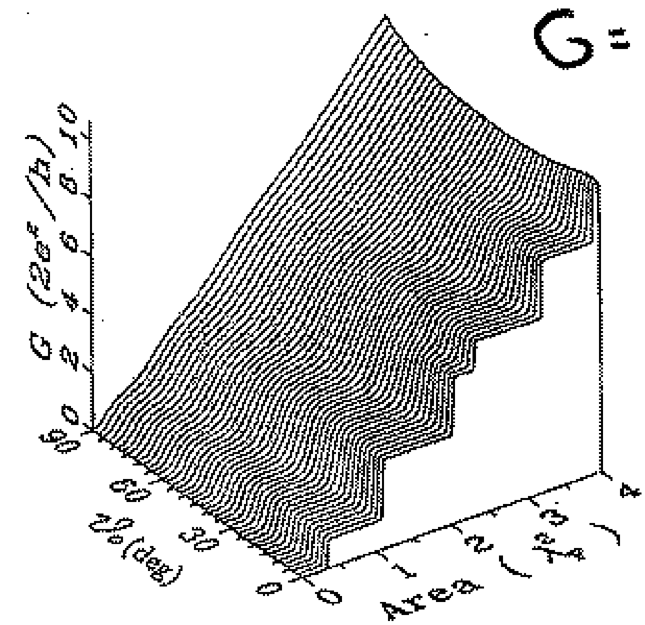
[Torres, Pascual, and Sáenz, PRB **49**, 16581 (1994)]



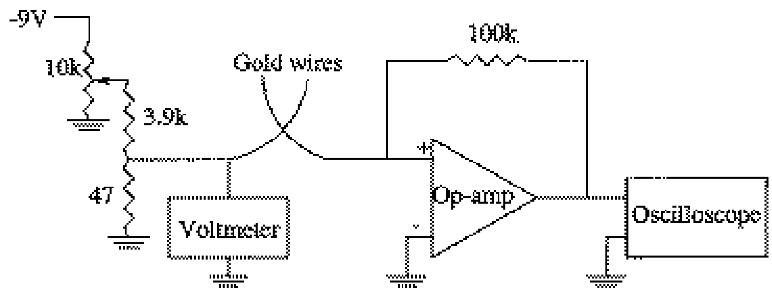
$f = k_F R$

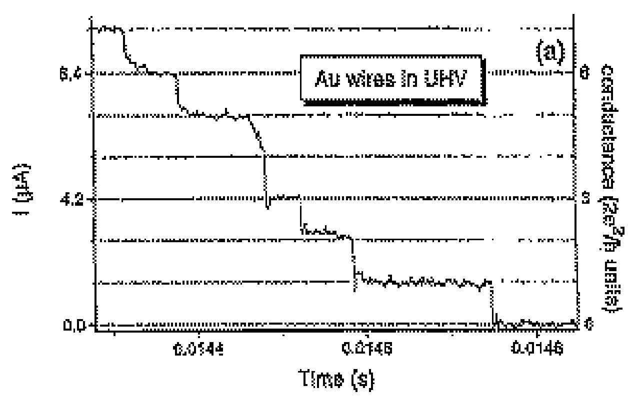
$$T_{\nu\nu'} = \frac{\delta_{\nu\nu'}}{1 + \exp\left(\frac{\pi}{f}\left[\varepsilon_\nu - m^2 - f^2 + 1/4\right]\right)}$$

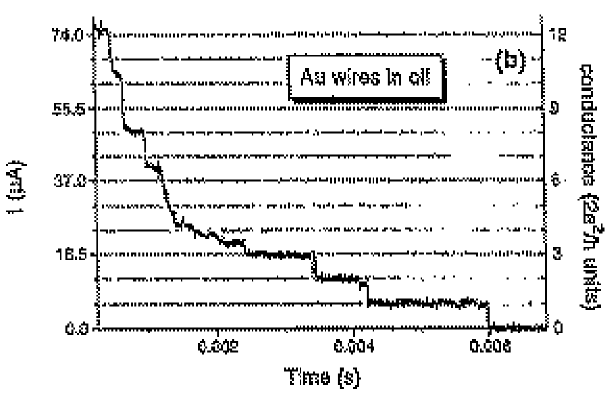
$$G = \frac{2e^2}{h} \sum_{\nu\nu'} T_{\nu\nu'}$$



Spontaneously occuring nanocontacts





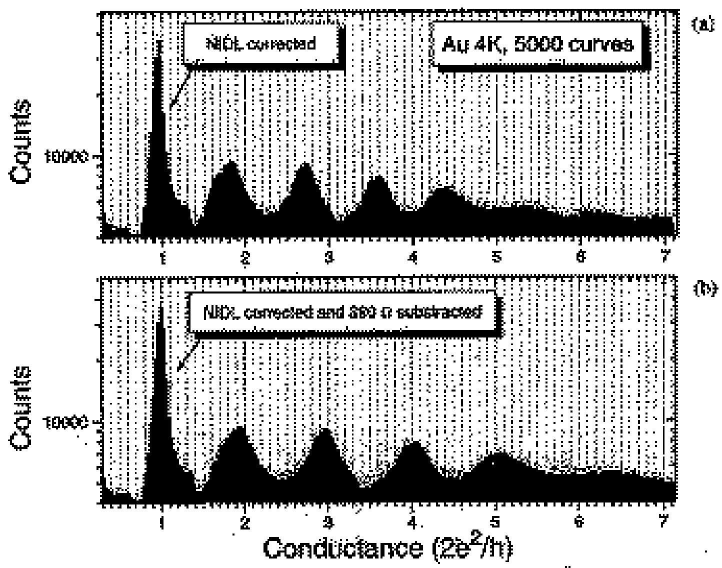


J. L. Costa-Krämer *et al.*, Surf. Sci. **342**, L1144 (1995)

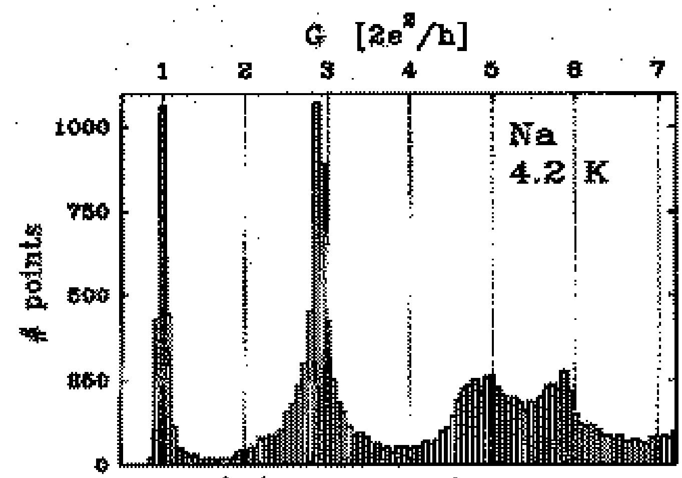
E. L. Foley *et al.*, Am. J. Phys. **67**, 389 (1999)

2. Quantum transport:

Conductance histograms



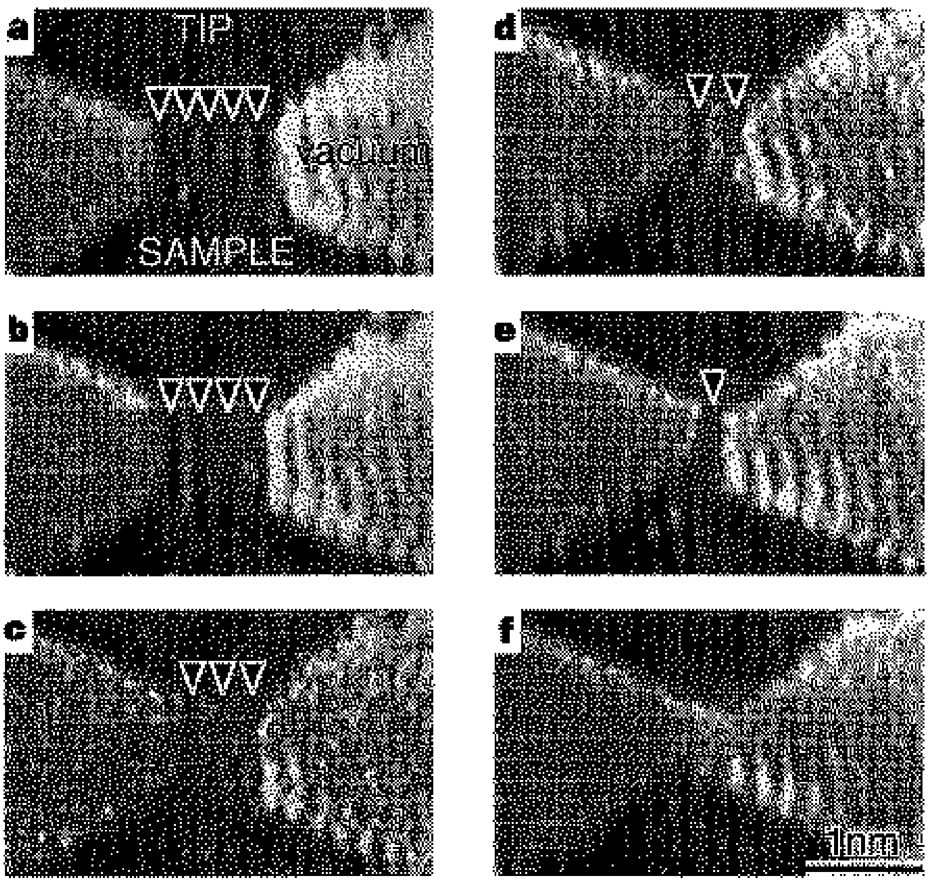
J. L. Costa-Krämer *et al.*, PRB **55**, 12910 (1997)



axial symmetry

J. M. Krans *et al.*, Nature **375**, 767 (1995)

Electron microscope video of an atomic-scale gold contact breaking

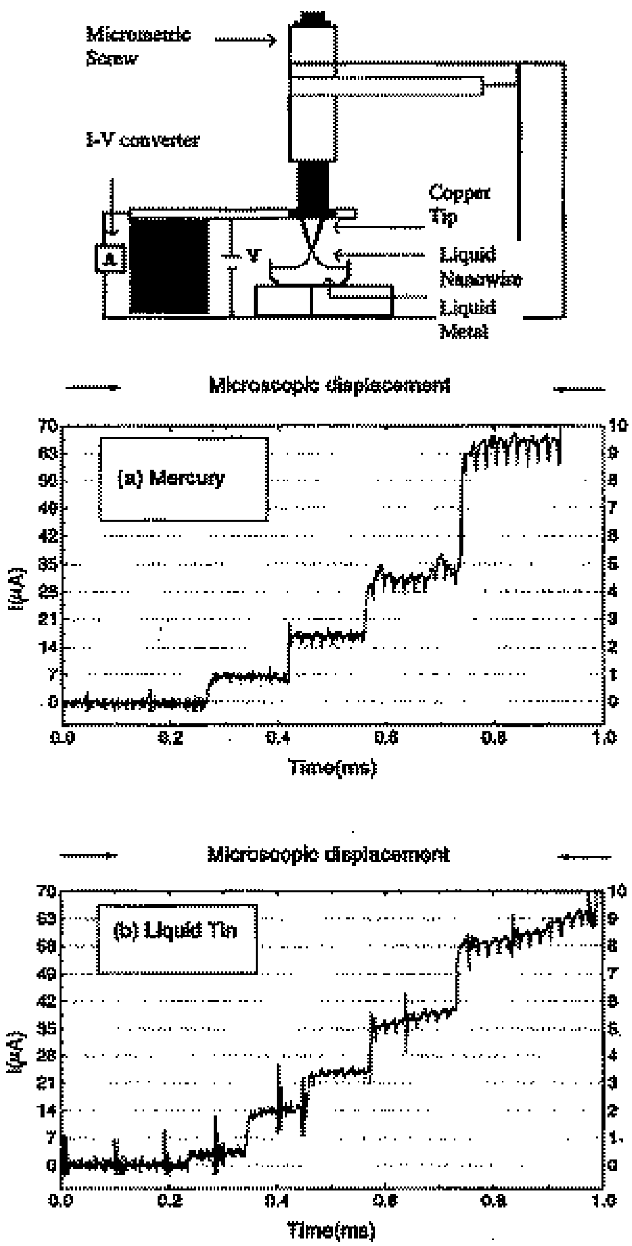


$G = 2G_0$

$G = G_0$

H. Ohnishi *et al.*, Nature **395**, 780 (1998)

Liquid metal nanowires



J. L. Costa-Krämer *et al.*, PRB **55**, 5416 (1997)

Quantum Hall Effect

$$\frac{h}{e^2} = 25.8\,k\Omega$$

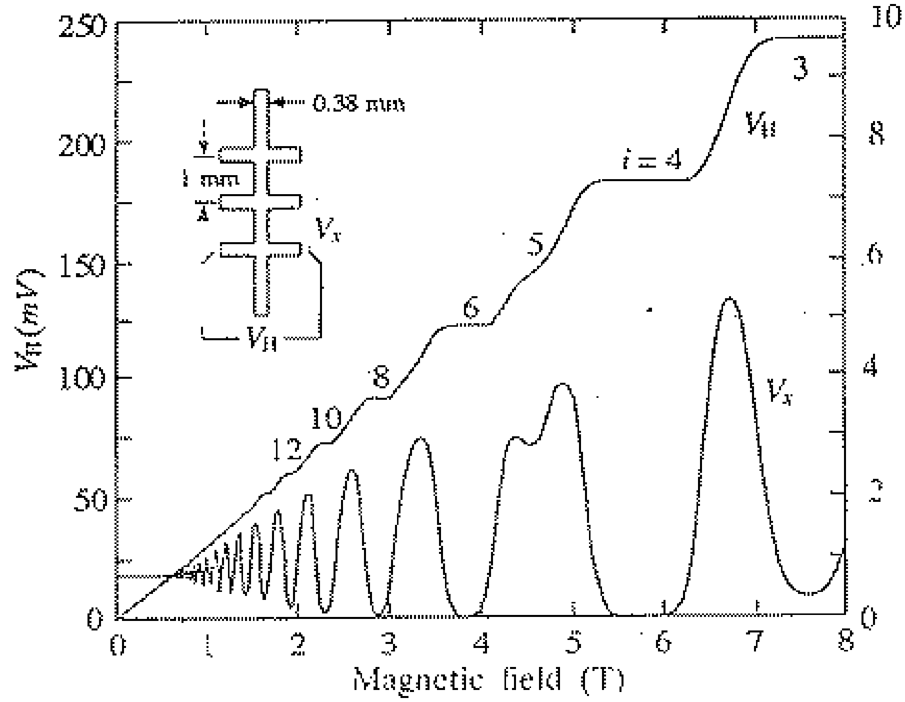


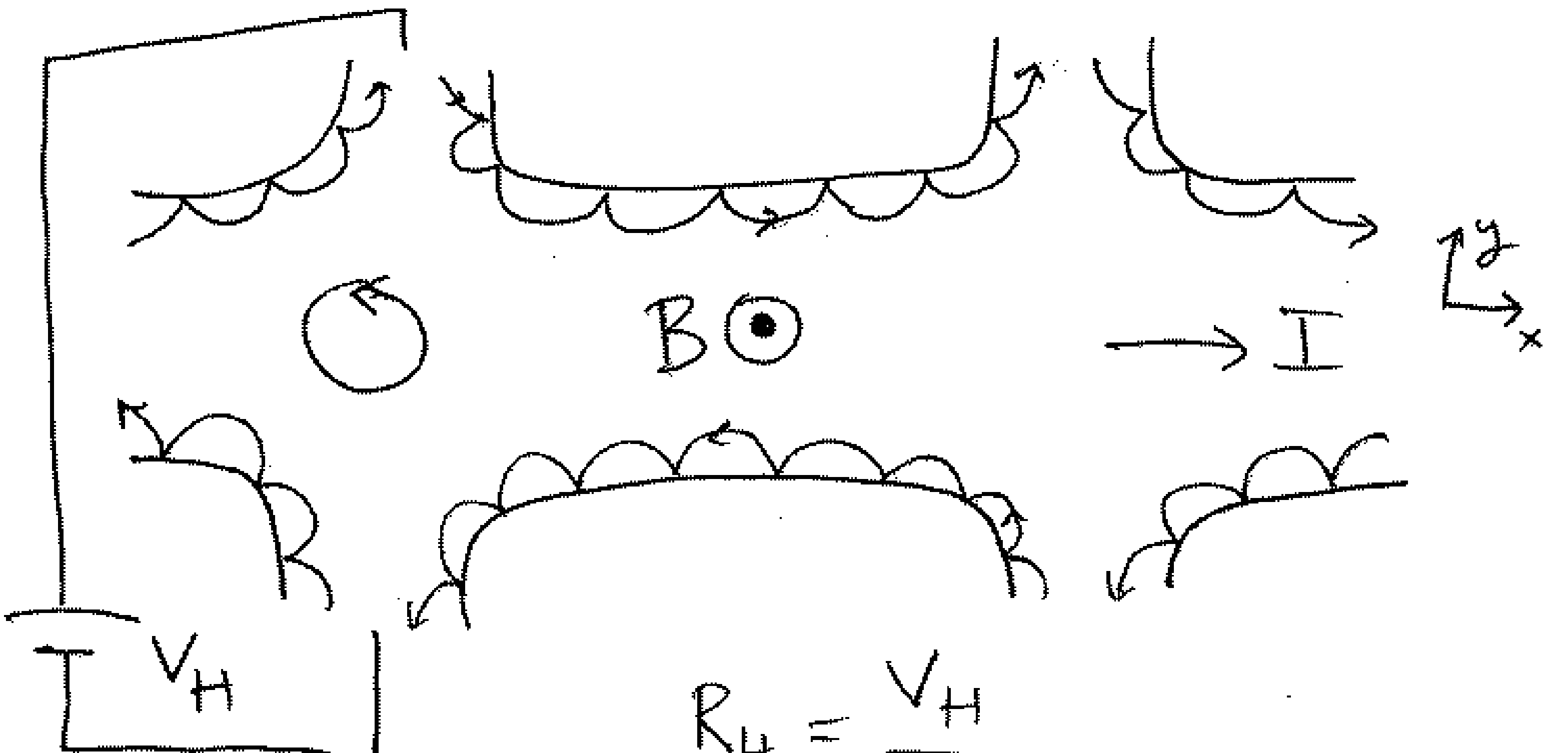
Fig. 1.4.2. Measured longitudinal and transverse voltages for a modulation-doped GaAs film at $T = 1.2$ K ($I = 25.5$ μA). Reproduced with permission from Fig. 1 of M. E. Cage, R. F. Dziuba and B. F. Field (1985), *IEEE Trans. Instrum. Meas.* IM-34, 301. © 1985 IEEE

$$R_H = \frac{h}{\nu e^2}$$

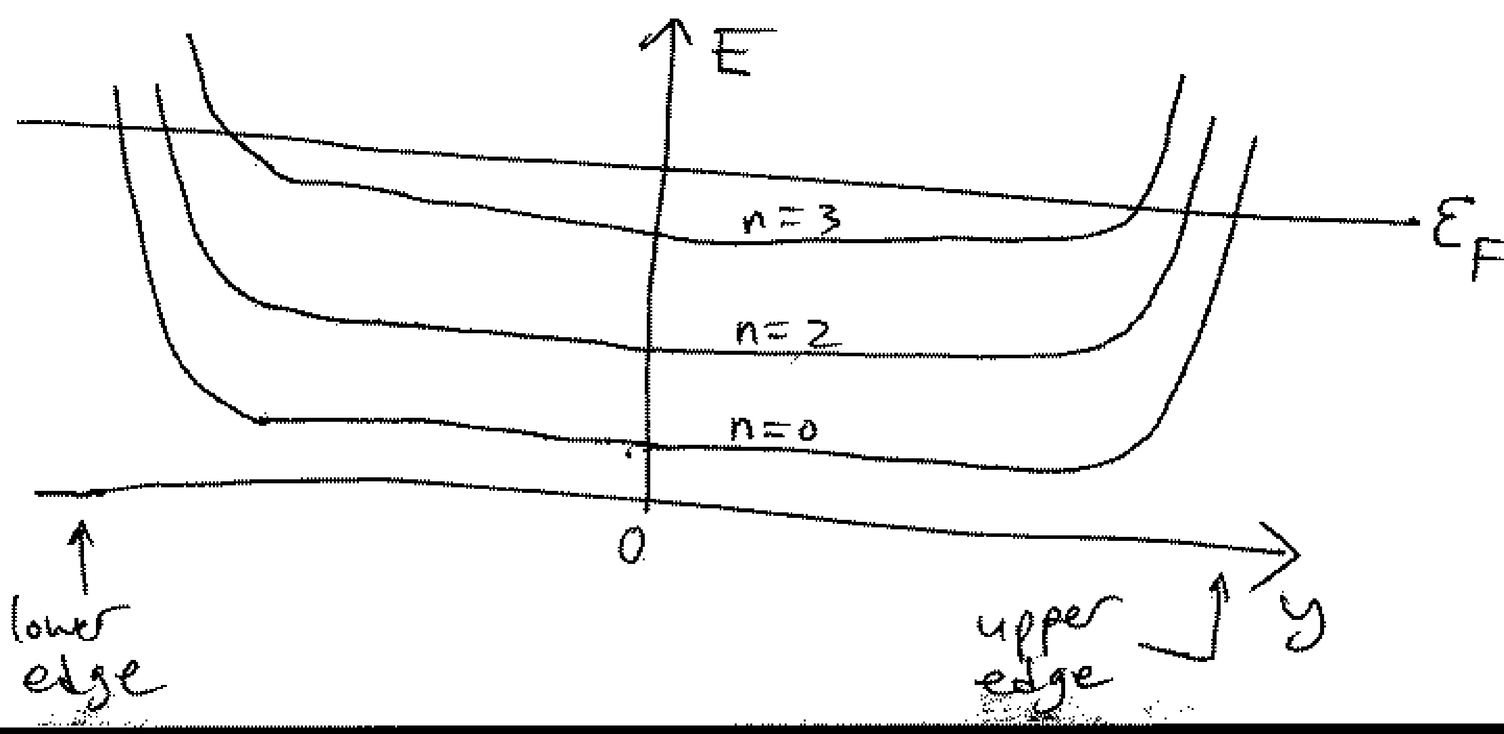
classically: $R_H = \frac{B}{ne}$

Quantum Hall effect

Current carried by quantized "edge states":



$$R_H = \frac{V_H}{I}$$



Group velocity

$$V_x^{(n)} = \frac{1}{\hbar} \frac{\partial E_n}{\partial k_x}$$

Recall $\quad y_0 = \frac{\hbar c k_x}{eB}$

$$V_x^{(n)} = \frac{1}{\hbar} \frac{\hbar c}{eB} \frac{\partial E_n}{\partial y_0}$$

$$E_n(y_0) \simeq \hbar \Omega \left(n + \tfrac{1}{2}\right) + U(y_0)$$

$U(y) =$ confining potential

$$V_x^{(n)} \simeq \frac{c}{eB} \frac{\partial U}{\partial y_0}$$

approx. indep. of quantum # n

states at upper edge travel right, lower go left.

These edge states form ideal one-dimensional channels. The # of channels is equal to the # of Landau levels lying below $\varepsilon_F \Longrightarrow \nu$ channels.

$$R_H = \frac{V_H}{I} = \frac{h}{\nu e^2}$$

Recall degeneracy $N_n = \frac{eBA}{hc}$

$$\nu = \mathrm{Int}\left\{\frac{N}{N_n}\right\} = \mathrm{Int}\left\{\rho_{2D}\frac{hc}{eB}\right\}$$

Classical limit $\nu \approx \rho_{2D}\frac{hc}{eB}$

$$R_H = \frac{h}{\nu e^2} \approx \frac{B}{\rho_{2D}ec} \checkmark$$