Physics 472 lecture 7

Addition of angular momenta

Consider two spin-1/2 particles, say the proton and neutron inside a deuteron. What can we say about the total angular momentum, assuming the orbital angular momentum is zero?

Let $\vec{S}_1$ and $\vec{S}_2$ be the spin operators for the two particles. $[\vec{S}_1, \vec{S}_2] = 0$ since they refer to different particles.

Since there are two linearly independent states for each spin, there are four linearly indep. states for two spins. Choosing the z-axis as the axis of quantization, we can write these states as

$$(1) \quad |\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle$$

where the first symbol refers to the eigenvalue of S_{1z} and the second to the eigenvalue of S_{2z}.

The total spin angular momentum of the two particles is

$$\vec{S} = \vec{S}_1 + \vec{S}_2 .$$

$\vec{S}$ obeys canonical commutation

relations for angular momentum:

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}]$$

$$= [S_{1x}, S_{1y}] + [S_{2x}, S_{2y}]$$

$$= i\hbar (S_{1z} + S_{2z}) = i\hbar S_z,$$

etc. Thus the eigenstates of $\vec{S}$ may be chosen as follows:

$$\vec{S}^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle$$

$$S_z |s, m\rangle = \hbar m |s, m\rangle$$

$$S_{\pm} |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)}\, |s, m \pm 1\rangle$$

with $-s \leq m \leq s$. ($2s+1$ values)

s may be an integer or half-odd integer. The allowed values of m differ by integers.

The basis states (1) are eigenstates of S_z:

$$S_z|\uparrow\uparrow\rangle = \left(\frac{\hbar}{2}+\frac{\hbar}{2}\right)|\uparrow\uparrow\rangle = \hbar\,|\uparrow\uparrow\rangle$$

$$S_z|\uparrow\downarrow\rangle = \left(\frac{\hbar}{2}-\frac{\hbar}{2}\right)|\uparrow\downarrow\rangle = 0$$

$$S_z|\downarrow\uparrow\rangle = \left(-\frac{\hbar}{2}+\frac{\hbar}{2}\right)|\downarrow\uparrow\rangle = 0$$

$$S_z|\downarrow\downarrow\rangle = \left(-\frac{\hbar}{2}-\frac{\hbar}{2}\right)|\downarrow\downarrow\rangle = -\hbar\,|\downarrow\downarrow\rangle.$$

The eigenvalues of S_z for the system of two particles are clearly 0 and $\pm\hbar$. We expect then to find a triplet of states with $S_z = -\hbar, 0, +\hbar$, corresponding to $S=1$ and a singlet with $S_z=0$, corresponding to $S=0$.

The state $|\uparrow\uparrow\rangle$ cannot have $S=0$, therefore it must have $S=1$, i.e.,

$$\vec{S}^2|\uparrow\uparrow\rangle = \hbar^2 S(S+1)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle.$$

To prove this, note that

$$\vec{S}^2 = (\vec{S}_1+\vec{S}_2)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\vec{S}_1\cdot\vec{S}_2$$

$$= \frac{3}{2}\hbar^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+},$$

where
$$S_{1\pm} = S_{1x} \pm i\,S_{1y},$$
$$S_{2\pm} = S_{2x} \pm i\,S_{2y}.$$

$$\Rightarrow \vec{S}^2|\uparrow\uparrow\rangle = \left(\frac{3}{2}\hbar^2 + 2\,\frac{\hbar^2}{4}\right)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$

since the terms involving S_{1+} or S_{2+} give zero.

By the same argument, the state $|\downarrow\downarrow\rangle$ is also a state with $s=1$. To construct the state with $s=1$, $m=0$, we can act upon $|\uparrow\uparrow\rangle$ with S_-:

$$S_-|\uparrow\uparrow\rangle = S_-|1,1\rangle = \hbar\sqrt{2-1\cdot 0}\;|1,0\rangle$$
$$= \hbar\sqrt{2}\;|1,0\rangle$$

But $S_- = S_{1-} + S_{2-}$, so

$$S_-|\uparrow\uparrow\rangle = S_{1-}|\uparrow\uparrow\rangle + S_{2-}|\uparrow\uparrow\rangle$$
$$= \hbar|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle$$

$$\boxed{|1,0\rangle = \frac{S_-}{\hbar\sqrt{2}}|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}$$

The state with $S=0$, $m=0$ must be orthogonal to $|1,0\rangle$,

so

$$\boxed{|s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}.$$

singlet

The three states with $S=1$ are

$$|s=1, m=1\rangle = |\uparrow\uparrow\rangle$$

$$|s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|s=1, m=-1\rangle = |\downarrow\downarrow\rangle$$

triplet

Two spin-$1/2$'s added together give total angular momentum $S=1$ or $S=0$. One can think of the spins as being "parallel" in the $S=1$ state and "antiparallel" in the $S=0$ state:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right) = \frac{1}{2}\vec{S}^2 - \frac{3}{4}\hbar^2$$

$$\vec{S}_1 \cdot \vec{S}_2 \,|S=1, m\rangle = \left(\frac{1}{2} 2\hbar^2 - \frac{3}{4}\hbar^2\right)|S=1, m\rangle$$

$$= \frac{1}{4}\hbar^2 \,|S=1, m\rangle$$

$$\vec{S}_1 \cdot \vec{S}_2 \,|S=0, m=0\rangle = -\frac{3}{4}\hbar^2 \,|S=0, m=0\rangle$$

Note that the <u>singlet</u> spin wavefunction is <u>antisymmetric</u> under interchange of particles, while the three <u>triplet</u> spin wavefunctions are <u>symmetric</u> under interchange of particles.