Physics 472 lecture 8

Addition of angular momenta II

1) Spin-orbit coupling

$$\vec{J} = \vec{L} + \vec{S}$$

$$[\vec{L}, \vec{S}] = 0$$

$$\Rightarrow \quad [J_x, J_y] = i\hbar J_z \quad , \text{etc.}$$

$$[\vec{S}^2, \vec{J}] = 0 \quad , \quad [\vec{L}^2, \vec{J}] = 0$$

but $\quad [S_z, \vec{J}^2] \neq 0 \qquad [L_z, \vec{J}^2] \neq 0$

$$[J_z, \vec{J}^2] = [L_z + S_z, \vec{J}^2] = 0$$

$J_z = L_z + S_z$ is a good quantum #.

Eigenstates of total ang. momentum:

$$\vec{J}^2 |j\, m_j\rangle = \hbar^2 j(j+1)\, |j\, m_j\rangle$$

$$J_z |j\, m_j\rangle = \hbar m\, |j\, m_j\rangle$$

$$J_\pm |j\, m_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j \pm 1)}\, |j\, m_j \pm 1\rangle$$

$$J_\pm = J_x \pm i J_y$$

The states $|j\, m_j\rangle$ are linear combinations of the states

$$|l\, m_l\rangle |s\, m_s\rangle .$$

2) General problem: addition of $\vec{J}_1 + \vec{J}_2$

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \quad , \quad [\vec{J}_1, \vec{J}_2] = 0$$

For given quantum #s j_1 & j_2, there are $(2j_1+1)(2j_2+1)$ basis states of the form

$$|j_1 m_1\rangle |j_2 m_2\rangle \equiv |j_1 m_1 j_2 m_2\rangle$$

We would like to find the eigenstates of $\vec{J}$, which are linear combinations of these basis states. Because

$$[\vec{J}_1^2, \vec{J}] = 0 \quad \text{and} \quad [\vec{J}_2^2, \vec{J}] = 0,$$

the eigenstates of $\vec{J}$ are also eigenstates of $\vec{J}_1^2$, $\vec{J}_2^2$, and we may label them

$$|j_1 j_2 j m\rangle.$$

Multiplying by the unit operator in this subspace gives

$$|j_1 j_2 j m\rangle = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle \times |j_1 m_1 j_2 m_2\rangle$$

The Clebsch-Gordan coefficients

$$\langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle \equiv \langle j_1 m_1 j_2 m_2 | j m\rangle$$

give the desired linear combination.

Note that

$$\langle j_1 m_1 j_2 m_2 | J_z | j m\rangle = \hbar m \langle j_1 m_1 j_2 m_2 | j m\rangle$$
$$= \hbar (m_1 + m_2) \langle j_1 m_1 j_2 m_2 | j m\rangle$$

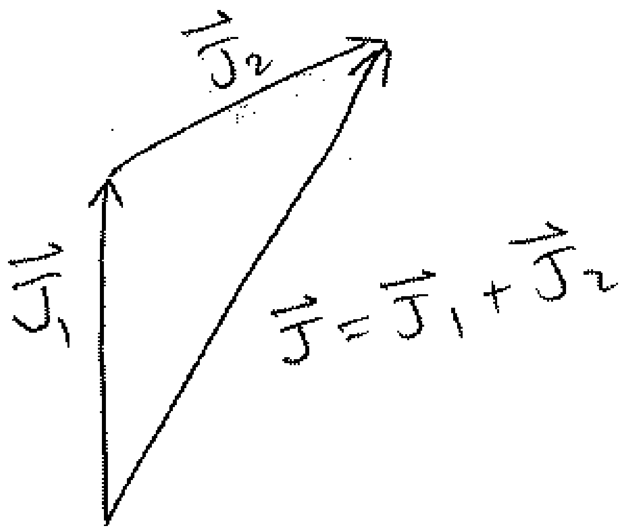
$\Rightarrow$ $\langle j_1 m_1 j_2 m_2 | j m\rangle = 0$ unless

$m = m_1 + m_2$

$$|j m\rangle = \sum_{m_1 + m_2 = m} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | j m\rangle$$

Another important property of the Clebsch - Gordon coefficients is that

$$\boxed{\begin{array}{l} \langle j_1 m_1 j_2 m_2 | j m \rangle \neq 0 \quad \text{only if} \\ |j_1 - j_2| \leq j \leq j_1 + j_2 \\ \underline{\text{triangle inequality}} \end{array}}$$



For a proof of the triangle rule, consider the possible values of m.

Note that $\max\{m\} = j_1 + j_2$.

Thus $\max\{j\} = \bar{j}_1 + \bar{j}_2$; otherwise a higher value of m could be generated by acting upon the state with $m = \bar{j}_1 + \bar{j}_2$ with J_+.

If $\bar{j}_1 + \bar{j}_2 \in \mathbb{Z}$ (integers), then $m_1 + m_2 \in \mathbb{Z}$, and $m = m_1 + m_2 \in \mathbb{Z}$ so $j \in \mathbb{Z}$. On the other hand, if $\bar{j}_1 + \bar{j}_2$ is a half-odd integer, e.g. $\bar{j}_1 = 1$, $\bar{j}_2 = 1/2$, then $m = m_1 + m_2$ is a half-odd integer, and j is a half-odd integer. In either case, the allowed values of j and m differ by integers.

To determine the minimum value of $j = |j_1 - j_2|$, one can use an inductive argument, as discussed in class. However, one can also use the Schwartz inequality:

$$-\sqrt{\langle \vec{J}_1^{\,2} \rangle \langle \vec{J}_2^{\,2} \rangle} \le \langle \vec{J}_1 \cdot \vec{J}_2 \rangle \le \sqrt{\langle \vec{J}_1^{\,2} \rangle \langle \vec{J}_2^{\,2} \rangle}$$

$$\langle \vec{J}_1 \cdot \vec{J}_2 \rangle = \frac{1}{2} \langle \vec{J}^{\,2} - \vec{J}_1^{\,2} - \vec{J}_2^{\,2} \rangle$$
$$= \frac{\hbar^2}{2} \left(J(J+1) - J_1(J_1+1) - J_2(J_2+1) \right)$$

$$\Rightarrow \boxed{J(J+1) \ge J_1(J_1+1) + J_2(J_2+1) - 2\sqrt{J_1(J_1+1)J_2(J_2+1)}}$$

Suppose, without loss of generality, that $J_1 \geq J_2$. 8

i) Check if $J = J_1 - J_2$ satisfies the inequality.

$$J(J+1) = (J_1 - J_2)(J_1 - J_2 + 1)$$

$$= J_1(J_1+1) + J_2(J_2+1) - 2J_2(J_1+1)$$

$$-J_2(J_1+1) \overset{?}{\geq} -\sqrt{J_1(J_1+1)J_2(J_2+1)}$$

$$J_2^2(J_1+1)^2 \leq J_1(J_1+1)J_2(J_2+1)$$

$$J_2(J_1+1) \leq J_1(J_2+1)$$

$$J_1J_2 + J_2 \leq J_1J_2 + J_1$$

$$J_2 \leq J_1 \checkmark$$

ii) <u>Check $J = J_1 - J_2 - 1$</u>

$$J(J+1) = (J_1 - J_2 - 1)(J_1 - J_2)$$
$$= J_1(J_1+1) + J_2(J_2+1) - 2J_1(J_2+1)$$

$$-J_1(J_2+1) \overset{?}{\geq} -\sqrt{J_1(J_1+1)J_2(J_2+1)}$$

$$J_1^2(J_2+1)^2 \leq J_1(J_1+1)J_2(J_2+1)$$

$$J_1(J_2+1) \leq (J_1+1)J_2$$

$$J_1J_2 + J_1 \leq J_1J_2 + J_2$$

$$J_1 \leq J_2 \quad \Rightarrow \text{contradiction}$$

Therefore

$$|j_1 - j_2| \leq j \leq j_1 + j_2 .$$

As a final test, let us verify the size of the $|j_1 j_2 j m\rangle$ Hilbert space:

$$N_H = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1)$$

$$= 2(j_1+j_2)+1 + \cdots + 2(j_1-j_2)+1$$

$$= \left[j_1+j_2-(j_1-j_2)+1\right] \times \frac{1}{2}\left[2(j_1+j_2)+1 + 2(j_1-j_2)+1\right]$$

$$= (2j_2+1)(2j_1+1)$$

This is the same # of states as in the $|j_1 m_1\rangle|j_2 m_2\rangle$ basis.