Physics 472 Lecture 9

Addition of angular momenta III

Recall $|jm\rangle = \sum_{m_1+m_2=m} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2 | jm\rangle$,

where $\langle j_1 m_1 j_2 m_2 | jm\rangle$ is a Clebsch-Gordon coefficient. Taking the inner product with $\langle jm|$ gives

$$\underline{1} = \sum_{m_1+m_2=m} |\langle j_1 m_1 j_2 m_2 | jm\rangle|^2 .$$

The sum of the squares of the Clebsch-Gordon coefficients is one.

1) $\underline{J_1 + S = 1/2}$

An important case, and a simple one, is the addition of $S=1/2$ to another angular momentum J_1.

A trivial case is $J_1 = 0$.

Then $\vec{J} = \vec{J}_1 + \vec{S}$, with $J = 1/2$.

For $J_1 \equiv \ell \neq 0$, there are exactly two allowed values

$$J = \ell \pm 1/2.$$

Let us construct the corresponding Clebsch-Gordon coefficients.

$$|J\ M\rangle = |\ell + 1/2, \ell + 1/2\rangle = |\ell\ \ell\ 1/2\ 1/2\rangle$$

$$|\ell\ m_\ell\ s\ m_s\rangle$$

(Clebsch-Gordon coeff. = $\underline{1}$)

$$J_- |\ell + 1/2, \ell + 1/2\rangle = \hbar\sqrt{(\ell+1/2)(\ell+3/2) - (\ell+1/2)(\ell-1/2)}$$

$$\times |\ell + 1/2, \ell - 1/2\rangle$$

$$= \hbar\sqrt{2\ell+1}\ |\ell + 1/2, \ell - 1/2\rangle$$

$$J_-|l\,l\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = J_{1-}|l\,l\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |l\,l\rangle J_{2-}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{l(l+1)-l(l-1)}\,|l\,l-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|l\,l\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2l}\,|l\,l-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|l\,l\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\Rightarrow \quad |l+\tfrac{1}{2},l-\tfrac{1}{2}\rangle = \sqrt{\frac{2l}{2l+1}}\,|l\,l-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\frac{1}{2l+1}}\,|l\,l\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

But we can also construct a linear combination of $|l\,l-1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$ and $|l\,l\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$ which is orthogonal to $|l+\tfrac{1}{2},l-\tfrac{1}{2}\rangle$. This can only be the state $|l-\tfrac{1}{2},l-\tfrac{1}{2}\rangle$.

$$|l-1/2, l-1/2\rangle = \sqrt{\frac{1}{2l+1}}\,|l\; l-1\; 1/2\; 1/2\rangle$$

$$- \sqrt{\frac{2l}{2l+1}}\,|l\, l\; 1/2\; -1/2\rangle$$

We can repeat the process to generate the states with lower values of total m. The result is

$$\langle l\; m \mp 1/2\;\; 1/2\; \pm 1/2 \,|\, l+1/2\;\; m\rangle = \sqrt{\frac{l \pm m + 1/2}{2l+1}}$$

$$\langle l\; m \mp 1/2\;\; 1/2\; \pm 1/2 \,|\, l-1/2\;\; m\rangle = \pm\sqrt{\frac{l \mp m + 1/2}{2l+1}}$$

The values of the Clebsch-Gordon coefficients for other values of J_1 and J_2 have been tabulated (see Griffiths for refs.).

2) Spin-Orbit interaction

Consider an electron traveling at velocity $\vec{v}$ through an electric field $\vec{E}$. In the rest frame of the electron, there is a magnetic field $\vec{B}' = -\frac{\vec{v}}{c} \times \vec{E}$.

This field will couple to the magnetic moment, leading to a correction to the Hamiltonian of the form

$-\vec{\mu} \cdot \vec{B}'$, where $\vec{\mu} = g\frac{(-e)}{2m_e c}\vec{S}$

and $g \simeq 2$ for an electron in vacuum.

There is a further relativistic correction due to the Thomas precession, which reduces the effect by a factor of two:

$$H_{s.o.} = -g\frac{e}{4m_ec^2}\vec{S}\cdot(\vec{V}\times\vec{E})$$

If the electric field is due to a central potential $V(r)$, as occurs to a first approximation in an atom, then

$$e\vec{E} = \nabla V = \frac{\vec{r}}{r}\frac{dV}{dr} \quad \text{and}$$

$$H_{s.o.} = \frac{g}{4m_ec^2}\vec{S}\cdot(\vec{r}\times\vec{V})\frac{1}{r}\frac{dV}{dr}$$

$$= \left(\frac{g}{4m_e^2c^2}\frac{1}{r}\frac{dV}{dr}\right)\vec{L}\cdot\vec{S}$$

The eigenstates of $\vec{J} = \vec{L} + \vec{S}$ are eigenstates of $\vec{L} \cdot \vec{S}$:

$$\vec{L} \cdot \vec{S} \, |\ell s j m\rangle = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) |\ell s j m\rangle$$

$$= \frac{\hbar^2}{2}\Big(j(j+1) - \ell(\ell+1) - s(s+1)\Big) |\ell s j m\rangle$$

i) For $j = \ell + 1/2$, we have

$$\frac{\vec{L} \cdot \vec{S}}{\hbar^2} |\ell s j m\rangle = \frac{\ell}{2} |\ell s j m\rangle$$

ii) For $j = \ell - 1/2$, we have

$$\frac{\vec{L} \cdot \vec{S}}{\hbar^2} |\ell s j m\rangle = -\frac{\ell - 1}{2} |\ell s j m\rangle$$

The eigenvalue is positive for $\vec{L}$ parallel to $\vec{S}$ and negative for $\vec{L}$ antiparallel to $\vec{S}$.