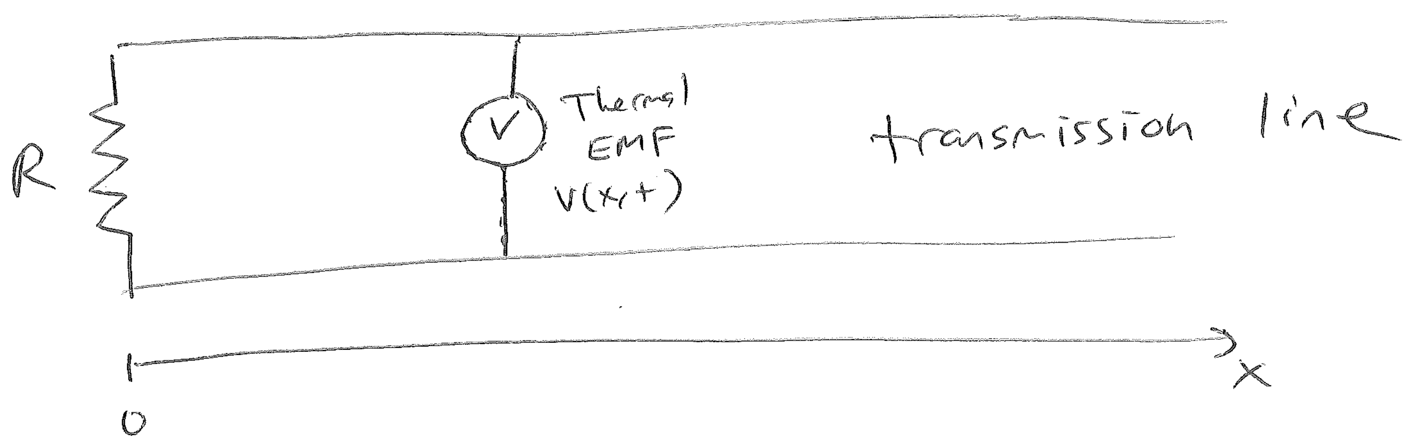
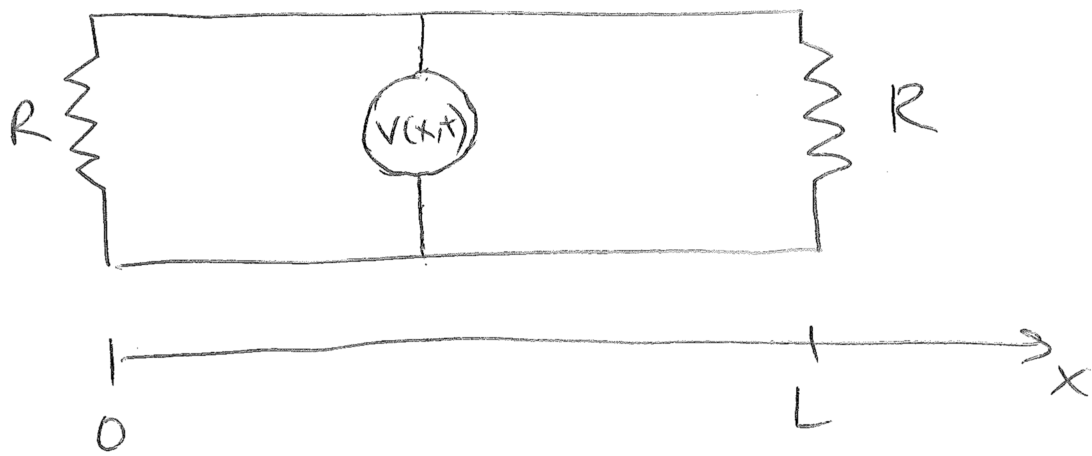
Phys 528, Lecture 20

1) Johnson-Nyquist noise or 1D Blackbody radiation



Consider a long transmission line, of impedance $Z = R$, terminated by a resistor R. The system is in thermal equilibrium at temperature T.
The transmission line is like a one-dimensional cavity for E+M waves. The resistor absorbs all waves incident upon it, and is therefore like a blackbody.



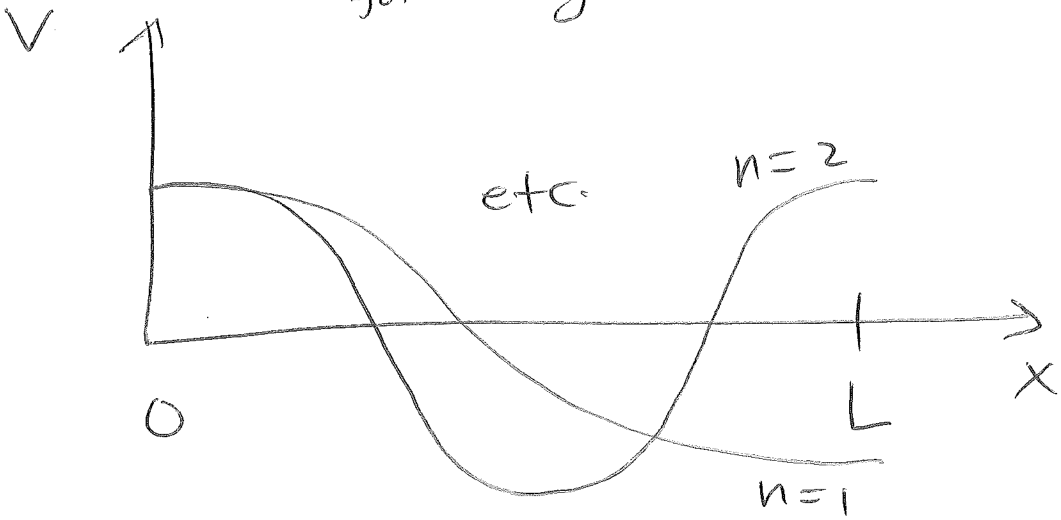
As far as our resistor on the left is concerned, the transmission line, terminated with a second resistor at $x=L$, looks the same as an infinite line, since R reflects nothing. The E+M waves in such a cavity satisfy Neumann boundary conditions:

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = \left.\frac{\partial V}{\partial L}\right|_{x=L} = 0$$

The solutions are of the form

$$V(x,t) = V_0 \sin(\omega t+\delta)\cos\left(\frac{n\pi x}{L}\right)$$

for a given t:



The wave equation for the transmission line is

$$\tilde{c}^2 \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2},$$

where $\tilde{c}$ is the speed of an E+M wave along the line. In general, $\tilde{c} \neq c$ due to the use of dielectrics between the two conductors.

$$\Rightarrow \quad \tilde{c}^2 \left(\frac{n\pi}{L}\right)^2 = \omega^2$$

$$\boxed{\omega_n = \frac{n\pi \tilde{c}}{L}}$$

The thermal ave. occupancy of mode n is

$$\langle s_n \rangle = \frac{1}{e^{\hbar\omega_n/T} - 1}$$

and the thermal average energy is

$$E = \sum_{n=1}^{\infty} \langle s_n \rangle \hbar\omega_n = \sum_{n=1}^{\infty} \frac{\hbar\omega_n}{e^{\hbar\omega_n/T} - 1}$$

$$\simeq \int_0^\infty dn \, \frac{\hbar\omega_n}{e^{\hbar\omega_n/T} - 1} = \frac{\hbar L}{\pi \tilde{c}} \int_0^\infty d\omega \, \frac{\omega}{e^{\hbar\omega/T} - 1}$$

$$\equiv L \int_0^\infty d\omega \; u_\omega$$

u_ω = energy per unit length per unit frequency in the transmission line.

$$\boxed{u_\omega = \frac{\hbar\omega/\pi\tilde{c}}{e^{\hbar\omega/T} - 1}}$$

The power absorbed by the resistor in the frequency range between ω and $\omega + d\omega$ is thus

$$P(\omega)\, d\omega = \frac{\tilde{c}\, u_\omega}{2}\, d\omega$$

(The factor of $\frac{1}{2}$ comes because half of the waves travel to the right and half travel left.)

$$P(\omega) = \frac{\hbar\omega / 2\pi}{e^{\hbar\omega/T} - 1}$$

Since the resistor is matched to the impedance of the transmission line, it absorbs all power incident on it. In thermal equilibrium, the resistor must emit a power equal to that which it absorbs. (Otherwise it would heat up or cool down.) This must be true in each frequency interval, since we could insert a narrow-band filter between the resistor and the transmission line.

∴ A resistor at temperature T must emit a power $\mathcal{P}(\omega) = \dfrac{\hbar\omega/2\pi}{e^{\hbar\omega/T} - 1}$ per unit frequency. The total power emitted by the resistor is

$$P = \int_0^\infty d\omega\, \mathcal{P}(\omega) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\hbar\omega}{e^{\hbar\omega/T} - 1}$$

let $\hbar\omega/T = x$

$$P = \frac{T^2}{h} \underbrace{\int_0^\infty dx \frac{x}{e^x - 1}}_{I} = \Gamma(2)\zeta(2) = \frac{\pi^2}{6}$$

$$I = \int_0^\infty dx \frac{xe^{-x}}{1 - e^{-x}} = \sum_{n=0}^\infty \int_0^\infty dx\, x e^{-x} (e^{-x})^n = \sum_{n=1}^\infty \underbrace{\int_0^\infty dx\, x e^{-nx}}_{\frac{1}{n^2}}$$

$$I = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

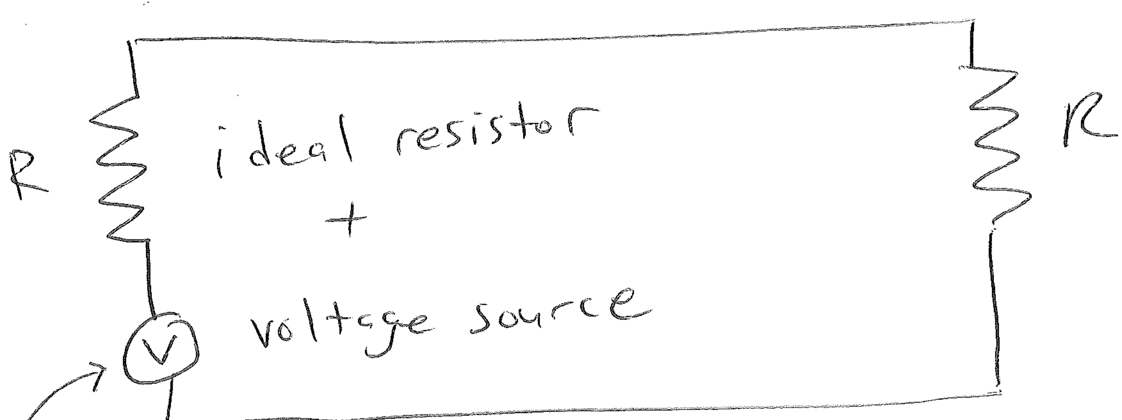
$$\Rightarrow \boxed{P = \frac{\pi^2}{6}\frac{T^2}{h}}$$

Current and voltage noise

$$P = \int_0^\infty d\omega\, \mathcal{P}(\omega) = \langle I^2 \rangle R$$

$$\left[\begin{array}{l} \langle I \rangle = 0 \text{ in equilibrium, but} \\ \langle I^2 \rangle > 0 \end{array} \right]$$

Equivalent circuit for noisy resistor:



This voltage source mimics the noise voltage in the real resistor.

From the equivalent circuit, it is clear that

$$V = 2RI$$

$$\langle I^2 \rangle R = \frac{\langle V^2 \rangle}{4R}$$

$$\boxed{\begin{aligned} \langle V^2 \rangle &= 4R \int_0^\infty d\omega\, \rho(\omega) \\ &\equiv \int_0^\infty J_+(\omega) \frac{d\omega}{2\pi} \end{aligned}}$$

$J_+(\omega)$ is the power spectrum of voltage fluctuations.

$$\boxed{J_+(\omega) = 4R \frac{\hbar\omega}{e^{\hbar\omega/T} - 1}}$$

Johnson-Nyquist noise

Classical limit $\hbar \to 0$
$(\hbar\omega \ll T)$

$$J_+(\omega) \simeq 4R \frac{\hbar\omega}{1 + \frac{\hbar\omega}{T} + \cdots - 1}$$

$$\boxed{J_+(\omega) \underset{T \gg \hbar\omega}{\simeq} 4RT}$$

This is an example of the "fluctuation-dissipation theorem."
The voltage fluctuations $\langle V^2 \rangle$ are proportional to the coefficient R describing dissipation when a current flows.

$$\langle V(t)V(t)\rangle \equiv \int_0^\infty J_+(\omega)\,\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} J(\omega)\,\frac{d\omega}{2\pi}$$

$$\langle V(t)\,V(t+\tau)\rangle = \int_{-\infty}^{\infty} J(\omega)\,e^{i\omega\tau}\,\frac{d\omega}{2\pi}$$

$$J(\omega) = \int_{-\infty}^{\infty} d\tau\,\langle V(t)\,V(t+\tau)\rangle\,e^{-i\omega\tau}$$

$$J_+(\omega) = 2J(\omega)$$

$$\langle V^2\rangle = \int_0^\infty \frac{d\omega}{2\pi} J_+(\omega) = 4R\int_0^\infty d\omega\,P(\omega)$$

$$= 4R\int_0^\infty \frac{d\omega}{2\pi}\,\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$J_+(\omega) = 4R\,\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$\int_{-\infty}^{\infty} d\tau\,\langle V(t)V(t+\tau)\rangle e^{-i\omega\tau} = 2R\,\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$J_+(\omega) \underset{k_BT \gg \hbar\omega}{\simeq} 4k_BT\,R$$

$$\langle I^2 \rangle R = \int_0^\infty d\omega\, P(\omega)$$

$$\langle I^2 \rangle = G\int_0^\infty d\omega\, P(\omega)$$

Let

$$\langle I(t) I(t+\tau) \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S(\omega)\, e^{i\omega\tau}$$

$$\langle I^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S(\omega) \equiv \int_0^\infty \frac{d\omega}{2\pi}\, S_+(\omega)$$

$$= G\int_0^\infty d\omega\, P(\omega)$$

$$S_+(\omega) = G\,\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

$$\underset{\hbar\omega \ll k_BT}{\simeq} G\,k_B T$$

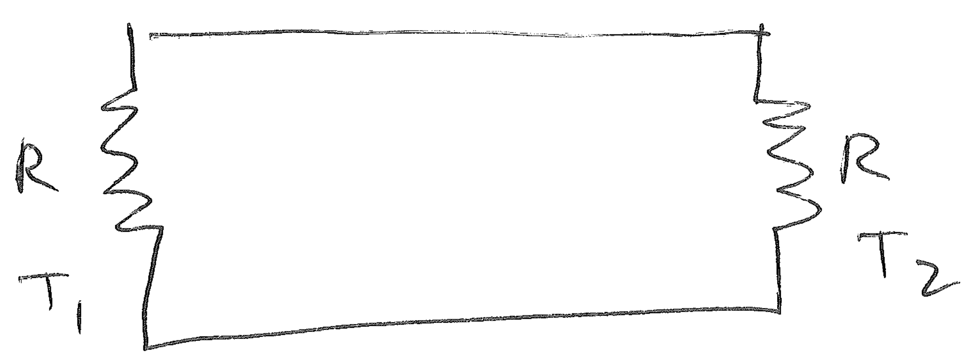
$$S_+(\omega) = 2S(\omega) = 2\int_{-\infty}^{\infty} d\tau \, \langle I(t) I(t+\tau) \rangle e^{-i\omega\tau}$$

$$\int_{-\infty}^{\infty} d\tau \, \langle I(t) I(t+\tau) \rangle e^{-i\omega\tau} = G \frac{\hbar\omega/2}{e^{\beta\hbar\omega} - 1}$$

<u>fluctuation-dissipation theorem</u>

→ Relates <u>equilibrium fluctuations</u> to dissipative response function G $(I = GV)$.

<u>Quantum of Thermal Conductance</u>



$$P_1 = \frac{\pi^2}{6}\frac{k_B^2 T_1^2}{h}, \qquad P_2 = \frac{\pi^2}{6}\frac{k_B^2 T_2^2}{h}$$

$$I^Q = P_1 - P_2 = \frac{\pi^2}{6}\frac{k_B^2}{h}\left(T_1^2 - T_2^2\right)$$

$$= \frac{\pi^2}{3}\frac{k_B^2}{h}\left(\frac{T_1+T_2}{2}\right)(T_1 - T_2)$$

$$= \left(\frac{\pi^2}{3}\frac{k_B^2}{h}T_0\right)(T_1 - T_2)$$

$$\boxed{K_0 = \frac{\pi^2}{3}\frac{k_B^2 T_0}{h} = \text{thermal conductance quantum}}$$

= thermal conductance of a perfect 1D channel