Physics 528 Lecture 19

(cont.)

Phenomenological theory of superconductivity

We have seen that a superconductor is characterized by a macroscopic condensate wavefunction for Cooper pairs

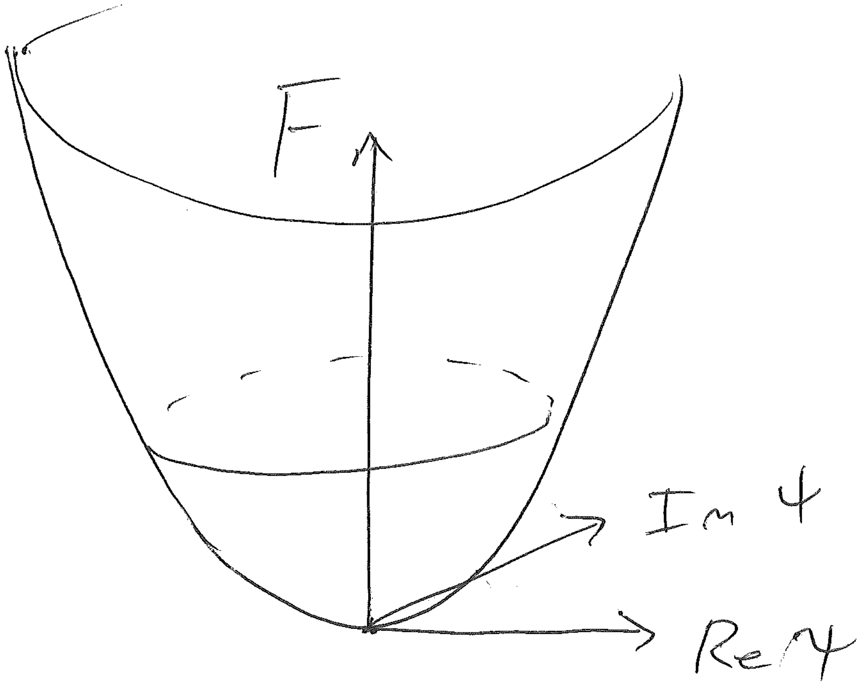
$$\Psi_s(\vec{r}) = \sqrt{n_s(\vec{r})}\, e^{i\theta(\vec{r})}.$$

Below T_c, $|\Psi_s(\vec{r})| > 0$; above T_c, $\Psi_s(\vec{r}) = 0$. $\Psi_s(\vec{r})$ can thus be considered the <u>order parameter</u> of the superconducting state. As $T \nearrow T_c$, $n_s \to 0$, so $\Psi_s(\vec{r})$ can be considered a small

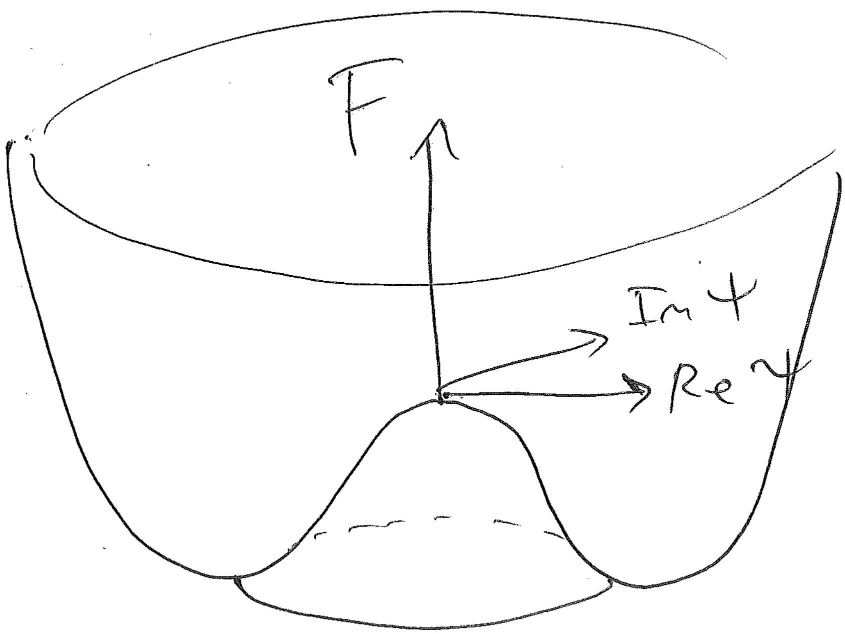
parameter for $T_c - T \ll T_c$.
In the absence of an external magnetic field, we can thus write the free energy of a superconductor as a Taylor series:

$$F_S = F_N + \int d^3r \left\{ \frac{\hbar^2}{2m^*} |\nabla \psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right\},$$

where higher-order terms have been omitted, and ~~terms~~ odd powers of ψ are excluded by symmetry. Here it is also assumed that $\psi(\vec{r})$ is a slowly varying function of $\bar{r}$. b is a positive constant, and



T > T_c



T < T_c

"Mexican hat"

a changes sign at $T = T_c$:

$$a(T) = \alpha(T - T_c).$$

In a homogeneous superconductor with no external field ψ is independent of $\vec{r}$:

$$F_s = F_N + aV|\psi|^2 + \frac{1}{2}bV|\psi|^4$$

equilibrium:

$$0 = \frac{\partial F_s}{\partial |\psi|^2} = aV + bV|\psi|^2$$

$$|\psi|^2 = -\frac{a}{b} = \frac{\alpha(T_c - T)}{b}.$$

Thus $n_s \to 0$ linearly as $T \nearrow T_c$.

Substituting this value of $|\psi|^2$

back into F_s gives

$$F_s - F_N = -V\frac{\alpha^2}{2b}(T_c - T)^2$$

$$= -\frac{V H_c^2}{8\pi}$$

$$\Rightarrow \quad H_c = \sqrt{\frac{4\pi\alpha^2}{b}}\,(T_c - T).$$

The entropy is given by

$$S = -\frac{\partial F}{\partial T}\Big|_{V,N}$$

$$S_s - S_N = V\frac{\alpha^2}{b}(T - T_c)$$

$$C_V = T\frac{\partial S}{\partial T}\Big|_{V,N}$$

$$(C_s - C_N)\Big|_{T_c} = V\frac{\alpha^2}{b}T_c \Rightarrow$$ discontinuity of specific heat

When a magnetic field is present, the field energy $\vec{B}^2/8\pi$ must be added to F.

Also $\nabla \to \nabla - \frac{iq}{\hbar c}\vec{A}$.

Thus

$$\boxed{F_S = F_N\Big|_{B=0} + \int d^3r \left\{ \frac{\vec{B}^2}{8\pi} + \frac{\hbar^2}{2m^*}\left|\left(\nabla - \frac{iq}{\hbar c}\vec{A}\right)\psi\right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right\}}$$

<u>Ginzburg-Landau free energy</u>

The differential equations governing the distribution of $\psi(\vec{r})$ and $\vec{A}(\vec{r})$ are now found by minimizing F_s with respect to the three independent functions ψ, ψ^*, and $\vec{A}$.

The complex quantity ψ is a set of two real quantities, so that ψ and ψ^* must be regarded as independent functions in the variation. Varying the integral with respect to ψ^* and integrating by parts, one finds

$$\delta F = \int d^3r \left\{ -\frac{\hbar^2}{2m^*}\left(\nabla - \frac{iq}{\hbar c}\vec{A}\right)^2 \psi + a\psi + b|\psi|^2\psi \right\} \delta\psi^*$$

$$+ \frac{\hbar^2}{2m} \oint \left(\nabla\psi - \frac{iq}{\hbar c}\vec{A}\psi\right)\cdot d\vec{S}\ \delta\psi^*.$$

Putting $\delta F = 0$ for arbitrary $\delta\psi^*$, we get

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)^2\psi + a\psi + b|\psi|^2\psi = 0.$$

Varying F_s with respect to ψ gives the complex conjugate equation, and therefore nothing new.

Varying F_s with respect to $\vec{A}$ gives (see advanced texts on electromagnetism):

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} ,$$

where

$$\vec{J} = -\frac{i q \hbar}{2m^*}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

$$- \frac{q^2}{m^* c} |\psi|^2 \vec{A}$$

$$= \vec{J}_s$$

($\vec{J}_N = 0$ in equilibrium!). Thus we indentify the parameter m^* as the mass of a cooper pair $= 2m_e$.

The boundary conditions on ψ and $\vec{A}$ are

$$\vec{n} \cdot \left(\frac{\hbar}{i} \nabla \psi - \frac{q}{c} \vec{A} \psi \right) = 0$$

and $\vec{B}$ = continuous.

The Ginzburg - Landau theory has two characteristic length scales:

i) Penetration depth In a weak field, one can neglect the dependence of $|\psi|^2$ on $\vec{B}$, and assume $|\psi|^2 = \alpha (T_c - T)/b$ throughout the sample. Then

$$\vec{J} = \left(\frac{q\hbar}{m^*} \nabla \theta - \frac{q^2}{m^* c} \vec{A} \right) |\psi|^2$$

and $\nabla \times \nabla \times \vec{B} = \frac{4\pi}{c} \nabla \times \vec{J}$

$$\Rightarrow \quad \nabla^2 \vec{B} = \frac{1}{\lambda^2} \vec{B}$$

$$\boxed{\lambda^2 = \frac{m^* c^2}{4\pi n_s q^2}}$$

same as London penetration depth

$$\boxed{\lambda = \sqrt{\frac{m^* c^2 b}{4\pi q^2 \alpha (T_c - T)}}}$$

ii) <u>Correlation length</u>

In the absence of a magnetic field, let's rescale ψ by its value for a homogeneous superconductor:

$$\Psi = \frac{\psi}{\psi_0}, \quad \psi_0 = \sqrt{\frac{\alpha (T_c - T)}{b}}$$

$$\Rightarrow \quad -\xi^2 D^2\psi - \psi + |\psi|^2\psi = 0$$

$$\boxed{\xi(T) = \frac{\hbar}{\sqrt{2m^*\alpha(T_c - T)}}}$$

The Ginzburg-Landau parameter is the temperature-independent ratio:

$$K = \frac{\lambda(T)}{\xi(T)} = \frac{m^* c}{\hbar |q|}\sqrt{\frac{b}{2\pi}}$$

ξ characterizes the length-scale over which ψ varies in inhomogeneous problems.

Type-II superconductors

If $K > \frac{1}{\sqrt{2}}$, it is possible for superconductivity to persist for $H > H_c$ (thermodynamic critical field). Superconductivity is then destroyed at a higher field H_{c_2} determined as follows.

Near H_{c_2} $|\psi|$ is small, so we can neglect the cubic term in the non-linear Schrödinger equation for ψ:

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\vec{A}\right)^2\psi = -a\psi.$$

we define H_{c_2} as the largest

field for which this equation has a nonzero solution. This equation is just the Schrödinger equation for a particle of charge q in a uniform magnetic field. The lowest eigenvalue is

$$\frac{\hbar \Omega_c}{2}, \quad \text{where} \quad \Omega_c = \frac{|q| H_{c_2}}{m^* c}.$$

Thus

$$-a = |a| = \frac{|q|\hbar H_{c_2}}{2m^* c}$$

$$\Rightarrow \quad H_{c_2} = \frac{2m^* c}{|q|\hbar} \alpha (T_c - T)$$

But $H_c = \sqrt{\frac{4\pi}{b}}\, \alpha\, (T_c - T)$.

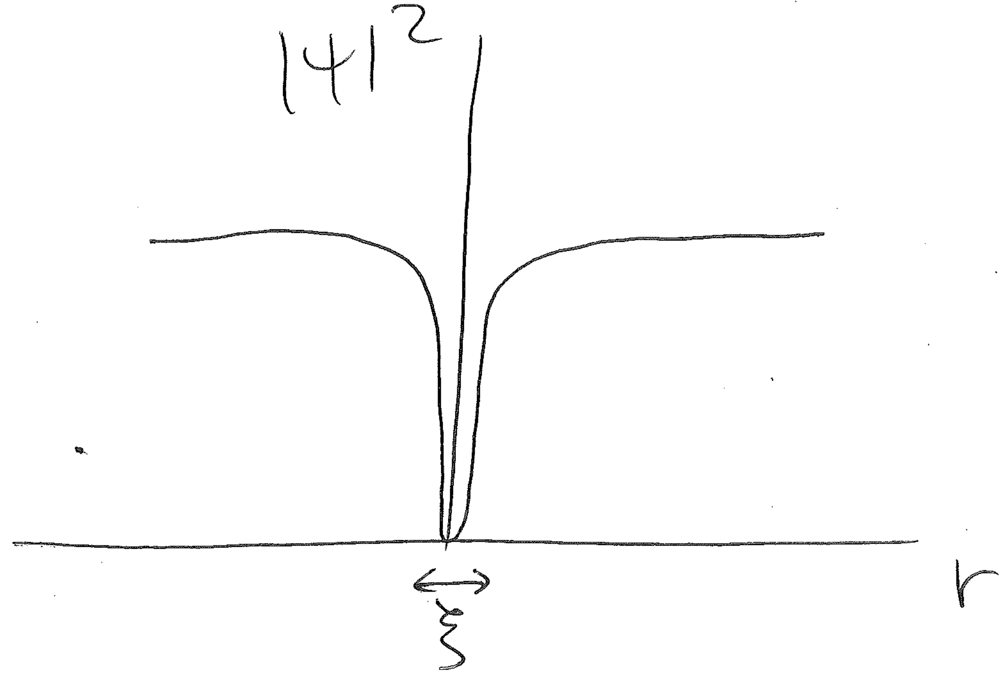
So $$\boxed{\frac{H_{c2}}{H_c} = \frac{2m^* c}{|q|\hbar}\sqrt{\frac{b}{4\pi}} = \sqrt{2}\,\kappa}$$

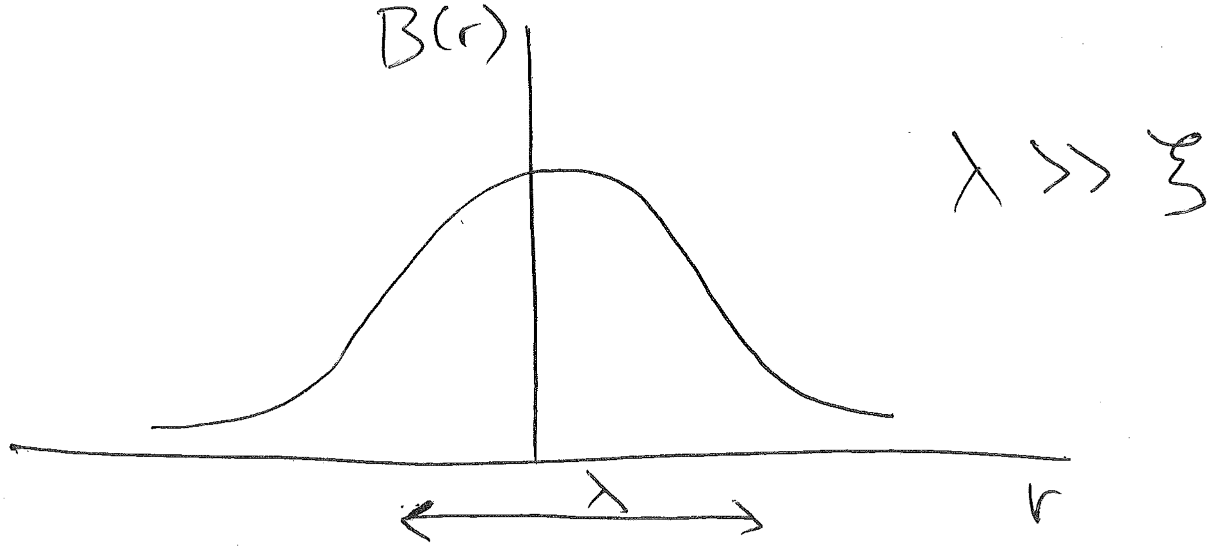
If $\kappa > \frac{1}{\sqrt{2}}$, then $H_{c2} > H_c$.

In type-II superconductors, the the interface between normal state and superconducting state has a <u>negative</u> surface tension.

The Meissner effect is not complete in type-II superconductors.

Above a <u>lower-critical field</u> H_{c1}, magnetic field penetrates the superconductor through nodes of the superconducting order parameter. Consider the case $\kappa \gg 1$, which is technologically relevant:





For $r > \xi$:

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} = \frac{4\pi q \hbar n_s}{m^* c} \left(\nabla\theta - \frac{q}{\hbar c} \vec{A} \right)$$

$$\vec{A} + \lambda^2 \nabla \times \vec{B} = \frac{\hbar c}{q} \nabla\theta$$

$$\oint \vec{A} \cdot d\vec{l} + \lambda^2 \oint \nabla \times \vec{B} \cdot d\vec{l} = \frac{\hbar c}{q} 2\pi s$$

For a large enough loop, $\nabla \times \vec{B} = 0$.

$$\oint \vec{A} \cdot d\vec{l} = \Phi = \frac{hc}{2e} s = \phi_0 s.$$

Thus the magnetic flux through such a <u>vortex</u> is quantized. Typically $s = 1$, so $\Phi = \phi_0$.

The lower-critical field is the minimum field to nucleate a vortex:

$$\phi_0 \approx \pi \lambda^2 H_{c_1}$$

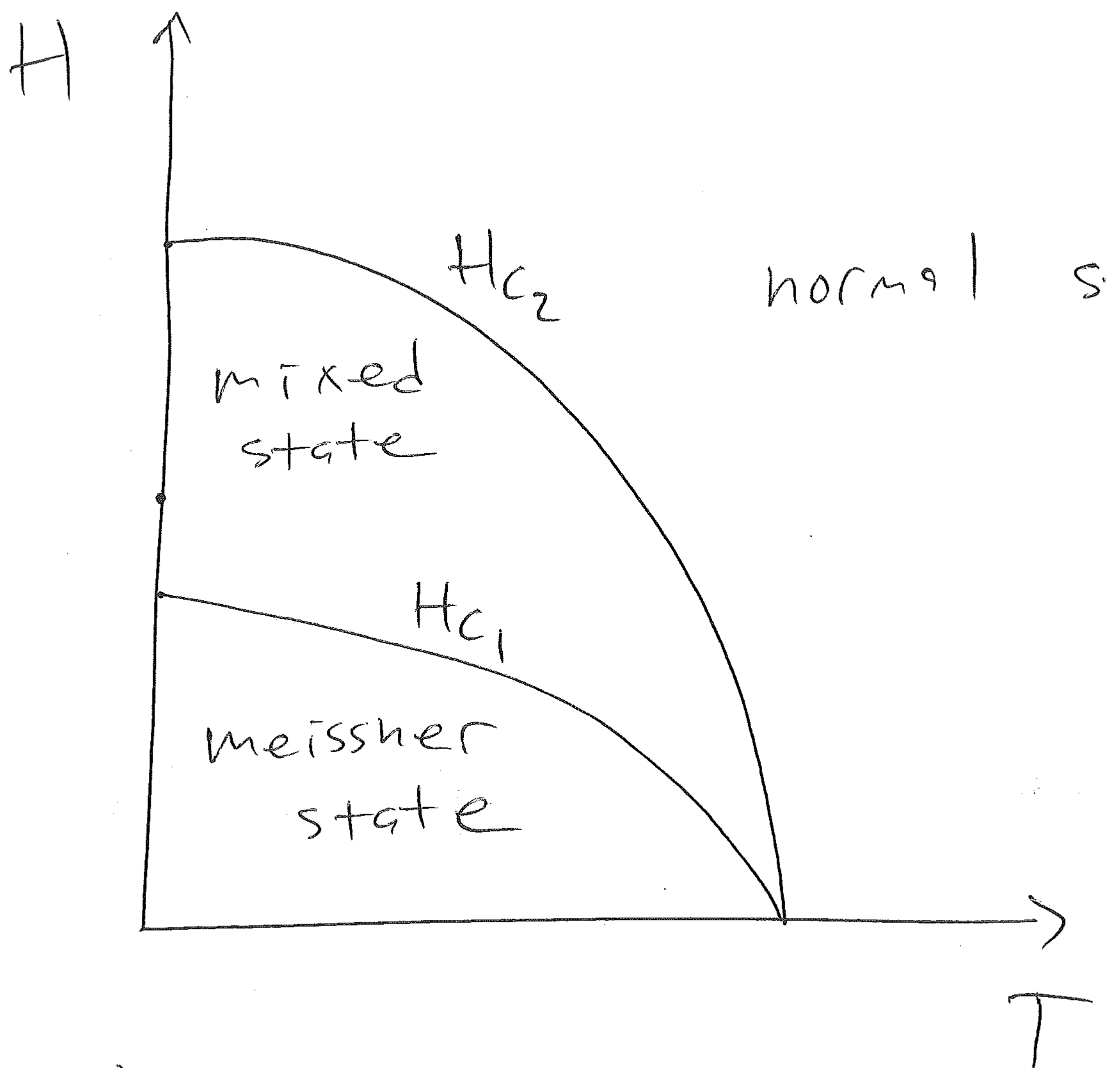
$$H_{c_1} \approx \frac{\phi_0}{\pi \lambda^2} \sim \frac{H_c}{\kappa}$$

One can also write

$$H_{c_2} = \frac{\phi_0}{2\pi \xi^2}.$$

For $H_{c_1} < H < H_{c_2}$, one has an Abrikosov lattice of magnetic flux tubes/vortices (show image). H_{c_2} corresponds to the minimum spacing between vortices $\sim \xi$.

Phase diagram



Pinning of vortices

by defects still gives perfect conductivity in mixed phase.

$\Rightarrow$ superconducting magnets.

critical exponents

Order parameter:

$$\psi = \sqrt{\frac{\alpha}{b}(T_c - T)} \sim |T_c - T|^{1/2}$$

$$\psi \sim |T_c - T|^{\beta} \qquad \beta = 1/2$$

Specific heat

$$(C_s - C_N)\big|_{T_c} = V \frac{\alpha^2}{b} T_c \qquad \text{discontinuity}$$

$$C_V \sim \begin{cases} C_s + \text{const.}\,(T_c - T)^{\alpha}, & T < T_c \\ C_N + \text{const.}\,(T - T_c)^{\alpha'}, & T > T_c \end{cases}$$

$$\alpha = \alpha' = 0$$

Critical field

$$H_c = \sqrt{\frac{4\pi\alpha^2}{b}}\,(T_c - T)^{\tau}, \quad \tau = 1$$

Ginzburg - Landau theory of Ferromagnetism

Order parameter $\vec{m}(\vec{r})$

magnetization

For $T_c - T \ll T_c$, $\frac{|\vec{m}|}{m_0} \ll 1$

$$F_{FM} = F_{para} + \int d^3r \left\{ a|\vec{m}|^2 + \frac{b}{2}|\vec{m}|^4 \cancel{+ c\nabla\vec{m}} - \vec{m}\cdot\vec{H} \right\}$$

$$0 = \frac{\delta F}{\delta \vec{m}} = 2a\vec{m} + 2b|\vec{m}|^2\vec{m} - \vec{H}$$

$a = \alpha(T - T_c)$, $b > 0$

For $\vec{H} = 0$,

$$|\vec{m}|^2 = \frac{\alpha(T_c - T)}{b}$$

$$|\vec{m}| \sim |T_c - T|^{1/2}, \qquad \underline{\beta = 1/2}$$

For $\vec{H} \neq 0$,

$$2a\,\vec{m} + 2b\,|\vec{m}|^2\,\vec{m} = \vec{H}$$

$$2a\,|\vec{m}| + 2b\,|\vec{m}|^3 = H$$

$$\left(2a + 6b|\vec{m}|^2\right)\frac{\partial m}{\partial H} = 1$$

$$\chi = \left.\frac{\partial m}{\partial H}\right|_{H=0} = \frac{1}{2a + 6b|\vec{m}|^2} = \begin{cases} \frac{1}{2a}, & T > T_c \\ \frac{1}{-4a}, & T < T_c \end{cases}$$

$$a = \alpha(T - T_c)$$

$$\chi \sim \begin{cases} |T - T_c|^{-\gamma'}, & T > T_c \\ |T_c - T|^{-\gamma}, & T < T_c \end{cases} \qquad \underline{\gamma = \gamma' = 1}$$

At $T = T_c$,

$$2b|\vec{m}|^3 = H$$

$$m \sim \left(\frac{H}{2b}\right)^{1/3} \sim H^{1/\delta}, \quad \underline{\delta = 3}$$

As for the sc, we find

$$\Delta F \equiv F_{FM} - F_{para} = V\left(a|\vec{m}|^2 + \frac{b}{2}|\vec{m}|^4\right), \quad \vec{H} = 0$$

$$= -V\frac{\alpha^2}{2b}(T_c - T)^2$$

$$S = -\frac{\partial F}{\partial T}\Big|_{V,\vec{H}}$$

$$S_{FM} - S_{para} = V\frac{\alpha^2}{b}(T - T_c)$$

$$C_V = T\frac{\partial S}{\partial T}\Big|_{V,\vec{H}}$$

$$(C_{FM} - C_{para})\Big|_{T_c} = V\frac{\alpha^2}{b}T_c \rightarrow \text{discontinuity}$$