Homework 4, Physics 528
Due in class Tuesday September 27

1) Kelvin's statement of the *Second Law* holds that no process is possible whose sole effect is to convert an amount of heat extracted from a reservoir at fixed temperature T entirely into work. Show that Kelvin's statement is violated for the case where the heat is extracted from a reservoir at absolute negative temperature. What is the maximum amount of work that can be generated by a heat engine operating between two reservoirs at fixed temperatures, one of which is negative?

2) Elasticity of polymers The thermodynamic identity for a one-dimensional system is

$$TdS = dE - FdL,$$

where F is the external force exerted on the line and dL is the extension of the line. It follows that

$$F = -T\left.\frac{\partial S}{\partial L}\right|_E .$$

The direction of the force is opposite to the conventional direction of the pressure.

We consider a polymeric chain of N links each of length ℓ, with each link equally likely to be directed to the right and to the left.

a) Show that the number of arrangements that give a head-to-tail length of $L = 2|s|\ell$ is

$$\Omega(N,s) + \Omega(N,-s) = \frac{2N!}{(N/2+s)!(N/2-s)!}.$$

b) For $|s| \ll N$, show that

$$S(L) = k_B \ln 2\Omega(N,0) - k_B L^2/2N\ell^2.$$

c) Show that the force at extension L is

$$F = Lk_BT/N\ell^2.$$

The force is proportional to temperature. The force arises because the polymer wants to curl up: the entropy is higher in a random coil than in an uncoiled configuration. Warming a rubber band makes it contract; warming a steel wire makes it expand.

3) The probability that a system in the grand canonical ensemble has exactly N particles is given by

$$P(N) = \frac{z^N Z_N(V,T)}{\mathcal{Z}(\mu, V, T)},$$

where $z = e^{\beta\mu}$. Verify this formula, and show that in the case of a classical ideal gas the distribution of particles among the members of a grand canonical ensemble is identically a Poisson distribution. Calculate $\langle (\Delta N)^2 \rangle$ for this system both from the general formula given in Lecture 9 and from the Poisson distribution, and show that the two results are the same.

4) Quantum dot

A quantum dot is an ultrasmall metallic or semiconducting island. Consider a quantum dot so small that it has only a single orbital of energy ε, which may be empty, occupied with an up-spin electron, occupied with a down-spin electron, or doubly occupied. If the orbital is doubly occupied, there is an additional Coulomb energy U.

State	Occupancy	Energy
0	0	0
$\uparrow$	1	ε
$\downarrow$	1	ε
$\uparrow\downarrow$	2	$2\varepsilon + U$

Table 1: States of a quantum dot.

The quantum dot can exchange energy and electrons with an electron reservoir at temperature T and chemical potential μ.

a) Calculate the grand partition function $\mathcal{Z}$ of the quantum dot.

b) Calculate the average number of electrons $\langle N \rangle$ on the quantum dot. Plot $\langle N \rangle$ versus μ for several values of T.

c) Calculate the mean-square fluctuations $\Delta N^2 = \langle (N - \langle N \rangle)^2 \rangle$. Plot ΔN^2 versus μ for several values of T.