Phys. 528 Lecture 16

Fermi gas II

1) Entropy

$$C_V = T \frac{\partial S}{\partial T}\Big|_{V,N}$$

$$S(T) = \underset{\substack{\| \\ 0}}{S(0)} + \int_0^T \frac{C_V(T')}{T'}\, dT'$$

$$S(T) \underset{T \ll T_F}{\simeq} \frac{\pi^2}{2} N k_B \frac{k_B T}{\varepsilon_F} \qquad (\text{3D Fermi gas})$$

In general,

$$S = -k_B \,\mathrm{Tr}\{\hat{\rho} \ln \hat{\rho}\} = -k_B \langle \ln \hat{\rho} \rangle .$$

For a nonequilibrium steady-state,
$\hat{\rho}$ is diagonal in energy:

$$\hat{\rho} = \prod_{\otimes \varepsilon} \hat{\rho}_\varepsilon$$

$$\ln \hat{\rho} = \int d\varepsilon\, g(\varepsilon) \ln \hat{\rho}_\varepsilon$$

$$\rho_\varepsilon = \begin{pmatrix} 1-f(\varepsilon) & 0 \\ 0 & f(\varepsilon) \end{pmatrix}$$

$$\langle \ln \hat{\rho}_\varepsilon \rangle = (1-f(\varepsilon))\ln(1-f(\varepsilon)) + f(\varepsilon)\ln f(\varepsilon)$$

$$\boxed{S = -k_B \int d\varepsilon\, g(\varepsilon)\left[f(\varepsilon)\ln f(\varepsilon) + (1-f(\varepsilon))\ln(1-f(\varepsilon))\right]}$$

True for an arbitrary nonequilibrium steady-state distribution $f(\varepsilon)$ for independent fermions (no 2-body interactions).

In equilibrium, $f(\varepsilon) = \dfrac{1}{e^{\beta(\varepsilon-\mu)}+1}$

$$1-f(\varepsilon) = \frac{e^{\beta(\varepsilon-\mu)}}{e^{\beta(\varepsilon-\mu)}+1} = e^{\beta(\varepsilon-\mu)} f(\varepsilon)$$

$$S = -k_B \int d\varepsilon\, g(\varepsilon)\left[f(\varepsilon)\ln\frac{f}{1-f} - \ln\left(1+e^{-\beta(\varepsilon-\mu)}\right)\right]$$

$$S = -k_B \int d\varepsilon\, g(\varepsilon)\left[-\beta(\varepsilon-\mu)f(\varepsilon) - \ln\left(1+e^{-\beta(\varepsilon-\mu)}\right)\right]$$

$$S = \frac{E}{T} - \frac{\mu N}{T} - \frac{\Omega}{T}; \quad \Omega = E - TS - \mu N$$ ✓

2) Magnetism of a Fermi gas

i) Pauli paramagnetism

In free space, the electron has an intrinsic magnetic moment

$$\vec{\mu} = -g \frac{e}{2mc} \vec{S}, \quad \text{where } g \simeq 2.00$$

$$|\vec{\mu}| = \frac{e}{mc} \frac{\hbar}{2} = \frac{e\hbar}{2mc} \equiv \mu_B = \text{Bohr magneton}$$

Neglecting orbital effects for now, we expect that a gas of electrons will be paramagnetic due to the Zeeman effect.

$$\mathcal{Z} = \sum_{N_\uparrow} \sum_{N_\downarrow} \sum_{s} e^{-\beta (E_s + \mu_B (N_\uparrow - N_\downarrow) B - \mu (N_\uparrow + N_\downarrow))},$$

where $E_z = -\vec{\mu} \cdot \vec{B} = -\mu_z B$ for $\vec{B} = B\hat{z}$

$$= \frac{e}{mc} S_z B$$

$$= \begin{cases} +\mu_B B, & S_z = +\hbar/2 \\ -\mu_B B, & S_z = -\hbar/2 \end{cases}$$

The total magnetic moment of the system is

$$M_z = -\mu_B N_\uparrow + \mu_B N_\downarrow = \frac{1}{\beta} \frac{\partial \ln \mathcal{Z}}{\partial B_z}$$

$$\Omega = -k_B T \ln \mathcal{Z}$$

$$M_z = -\frac{\partial \Omega}{\partial B_z}$$

For independent fermions, $\Omega = \Omega_\uparrow + \Omega_\downarrow$

$$\Omega_\sigma = -k_B T \int d\varepsilon \, g_\sigma(\varepsilon) \ln\left(1 + e^{-\beta(\varepsilon + \sigma \mu_B B - \mu)}\right)$$

$\sigma = \pm 1$ for spin up (down)

$$M = -\mu_B \langle N_\uparrow - N_\downarrow \rangle$$

$$= -\mu_B \int d\varepsilon \, \frac{g(\varepsilon)}{2} \left[f_\uparrow(\varepsilon) - f_\downarrow(\varepsilon) \right]$$

$$= -\frac{\mu_B}{2} \int d\varepsilon \, g(\varepsilon) \left[\frac{1}{e^{\beta(\varepsilon + \mu_B B - \mu)} + 1} - \frac{1}{e^{\beta(\varepsilon - \mu_B B - \mu)} + 1} \right]$$

To linear order in B:

$$\boxed{\chi = \frac{1}{V}\frac{\partial M}{\partial B}\bigg|_{B=0} = \frac{\mu_B^2}{V}\int d\varepsilon\, g(\varepsilon)\left(-\frac{\partial f}{\partial \varepsilon}\right)}$$

magnetic susceptibility

$$\lim_{T\to 0}\chi = \frac{\mu_B^2}{V}g(\varepsilon_F) = \frac{3N}{2V}\frac{\mu_B^2}{\varepsilon_F}$$

$$\lim_{T\to\infty}\chi = \frac{\mu_B^2}{V}\int d\varepsilon\, g(\varepsilon)\left(-\frac{\partial}{\partial\varepsilon}e^{-\beta(\varepsilon-\mu)}\right)$$

$$= \beta\frac{\mu_B^2}{V}\underbrace{\int d\varepsilon\, g(\varepsilon)\, e^{-\beta(\varepsilon-\mu)}}_{\lim_{T\to\infty} N}$$

$$= \frac{N}{V}\frac{\mu_B^2}{k_B T}$$

$$\boxed{\chi_{para} = \frac{\mu_B^2}{V}\int d\varepsilon\, g(\varepsilon)\left(-\frac{\partial f}{\partial\varepsilon}\right) = \begin{cases} \frac{3}{2}n\frac{\mu_B^2}{\varepsilon_F}, & T\to 0 \\ n\frac{\mu_B^2}{k_B T}, & T\to\infty \end{cases}}$$

ii) Landau diamagnetism

Orbital energies quantized in a field $\vec{B} = B\hat{z}$:

$$\mathcal{E}_n(p_z) = \frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{p_z^2}{2m},$$

$$n = 0, 1, 2, \ldots$$

Landau levels

Degeneracy of each Landau level

$$g_n = L_x L_y \frac{eB}{hc} = \frac{\Phi}{\phi_0},$$

$$\phi_0 = \frac{hc}{e} = \text{normal flux quantum}$$

$$\Omega = -k_B T \int_{-\infty}^{\infty} \frac{L_z\, dp_z}{h} \sum_{n=0}^{\infty} g_n \ln\left(1 + e^{-\beta(\mathcal{E}_n(p_z) - \mu)}\right)$$

Let $\mu_B = \frac{e\hbar}{2mc}$ = Bohr magneton

For week fields,

$$\mu_B B \ll k_B T,$$

we can approximate the sum over n by an integral

$$\sum_{n=0}^{\infty} f(n+\tfrac{1}{2}) \simeq \int_0^{\infty} f(x)\,dx + \frac{1}{24} f'(0)$$

$$\Omega = -k_B T \frac{VeB}{h^2 c}\left[\int_0^{\infty} dx \int_{-\infty}^{\infty} dp_z \ln\left[1 + z e^{-\beta(2\mu_B B x + \frac{p_z^2}{2m})}\right] - \frac{1}{12}\beta\mu_B B \int_{-\infty}^{\infty} \frac{dp_z}{z^{-1} e^{\beta\frac{p_z^2}{2m}} + 1}\right]$$

First term is indep. of B. The second terms is

$$\Delta\Omega(B) = \frac{V}{12}(\mu_B B)^2 \frac{4\pi m}{h^3}\int_{-\infty}^{\infty} \frac{dp_z}{z^{-1} e^{\beta\frac{p_z^2}{2m}} + 1}$$

$$\Omega = \Omega_0(\mu) - \frac{1}{24}(2\mu_B B)^2 \frac{\partial^2 \Omega_0}{\partial \mu^2}$$

$$\Omega = \Omega_0(\mu) + \frac{1}{6}(\mu_B B)^2 \frac{\partial N}{\partial \mu}$$

At low temp.s, $\frac{\partial N}{\partial \mu} = \frac{3N}{2\epsilon_F}$

$$\chi = \frac{1}{V}\frac{\partial M}{\partial B} = -\frac{1}{V}\frac{\partial^2 \Omega}{\partial B^2} = -\frac{1}{3}\frac{\mu_B^2}{V}\frac{\partial N}{\partial \mu}$$

$$\chi \underset{T\to 0}{\simeq} -\frac{n}{2}\mu_B^2 = -\frac{1}{3}\chi_{para}$$

Holds quite generally as a function of T.

$$\chi = \chi_{para} + \chi_{dia} = \frac{2}{3}\chi_{para}$$

$$N = \int d\varepsilon \; g(\varepsilon) \, f(\varepsilon)$$

$$\frac{\partial N}{\partial \mu} = \int d\varepsilon \; g(\varepsilon) \left(-\frac{\partial f}{\partial \varepsilon} \right)$$

$$f(\varepsilon) \underset{T \to \infty}{\simeq} e^{-\beta(\varepsilon - \mu)}$$

$$-\frac{\partial f}{\partial \varepsilon} = \beta \, e^{-\beta(\varepsilon - \mu)} = \beta \, f(\varepsilon)$$

$$\frac{\partial N}{\partial \mu} \underset{T \to \infty}{\sim} \beta \int d\varepsilon \; g(\varepsilon) f(\varepsilon) = \beta N$$

$$\chi \underset{T \to \infty}{\sim} -\frac{1}{3} \mu_B^2 \frac{\partial N}{\partial \mu} = -\frac{\mu_B^2}{3} \frac{n}{k_B T}$$

$$= -\frac{1}{3} \chi_{para}$$

3) Extreme relativistic Fermi gas

$$\varepsilon = cp$$

$$\varepsilon_F = c p_F = \hbar c k_F$$

$$N = \frac{k_F^3 V}{3\pi^2} \qquad k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

$$E = \frac{2V}{8\pi^3}\int d^3k \, \hbar c|\vec{k}| \, f(\varepsilon)$$

$$E_0 = \frac{V\hbar c}{\pi^2}\int_0^{k_F} k^3 \, dk = \frac{V}{4\pi^2}\hbar c k_F^4$$

$$= \frac{3}{4}\frac{V k_F^3}{3\pi^2}\hbar c k_F = \frac{3}{4} N \varepsilon_F$$

$$= \frac{3}{4} N \hbar c \left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

Fermi pressure

$$P = -\left.\frac{\partial E}{\partial V}\right|_{N,S} = \frac{1}{4} N \hbar c \left(\frac{3\pi^2 N}{V}\right)^{1/3} \frac{1}{V}$$

$$P = \frac{E}{3V} \qquad PV = \frac{1}{3}E$$

$$\boxed{P = \frac{1}{4}(3\pi^2)^{1/3}\hbar c\left(\frac{N}{V}\right)^{4/3}}$$

This is to be contrasted with the result for a non-relativistic Fermi gas

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}$$

$\Rightarrow$ relativistic Fermi gas is easier to compress cf. stellar collapse.

$$\Omega = -k_B T \int_0^\infty d\varepsilon\, g(\varepsilon) \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$N(\varepsilon) = \frac{\left(\frac{\varepsilon}{\hbar c}\right)^3 V}{3\pi^2}$$

$$g(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{V\varepsilon^2}{\pi^2\hbar^3 c^3}$$

$$\Omega = -\frac{k_B T V}{\pi^2 \hbar^3 c^3}\int_0^\infty d\varepsilon\ \varepsilon^2 \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

integrating by parts

$$\Omega = -\frac{1}{3}\frac{V}{\pi^2 c^3 \hbar^3}\int_0^\infty d\varepsilon\ \frac{\varepsilon^3}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= -\frac{1}{3}E$$

Thus $PV = \frac{1}{3}E$ quite generally

for the extreme relativistic Fermi gas.