Phys 560A Take-home final

Solutions

a) $$G_0^r(E) = \frac{1}{E - \varepsilon + i0^+}$$

$$\Sigma_T(\omega) = V g(\omega) V^\dagger \qquad g = \text{GF of leads}$$

$$\Sigma_T^r(\omega) = \sum_\alpha \sum_{k\in\alpha} \frac{|V_\alpha|^2}{\omega - \epsilon_k + i0^+}$$

$$\Sigma_T^<(\omega) = i \sum_\alpha f_\alpha(\omega)\, \Gamma^\alpha(\omega),$$

where $$\Gamma^\alpha(\omega) = 2\pi |V_\alpha|^2 \sum_{k\in\alpha} \delta(\omega - \epsilon_k)$$

Note: $\sum_{k\in\alpha} \delta(\omega - \epsilon_k) \equiv D_\alpha(\omega)$ = density of states of lead α

$$\Sigma_T^r(\omega) = \sum_\alpha \sum_{k\in\alpha} |V_\alpha|^2 \left[\mathcal{P}\frac{1}{\omega - \epsilon_k} - i\pi\,\delta(\omega - \epsilon_k) \right]$$

$$= -\frac{i}{2} \sum_\alpha \Gamma^\alpha(\omega) + \mathrm{Re}\left\{ \cancel{\Sigma_T^r(\omega)} \right\} \to 0$$

neglect for simplicity

$$\Sigma_T^r(\omega) = -i\,\frac{\Gamma^1(\omega) + \Gamma^2(\omega)}{2} \equiv -i\,\bar{\Gamma}(\omega) \;(2$$

$$\boxed{G^r(\omega) = \frac{1}{\omega - \varepsilon + i\,\bar{\Gamma}(\omega)}}$$

$$G^<(\omega) = G^r \Sigma_T^< G^a = |G^r(\omega)|^2\,\Sigma_T^<(\omega)$$

$$\boxed{G^<(\omega) = \frac{i \sum_\alpha f_\alpha(\omega)\,\Gamma^\alpha(\omega)}{(\omega - \varepsilon)^2 + \bar{\Gamma}^2}}$$

b) $T_{12}(E) = \Gamma^{(1)}(E)\,G^r(E)\,\Gamma^{(2)}(E)\,G^a(E)$

$$\boxed{T_{12}(E) = \frac{\Gamma^{(1)}(E)\,\Gamma^{(2)}(E)}{(\omega - \varepsilon)^2 + [\bar{\Gamma}(E)]^2}}$$

Note: $T_{12}(\varepsilon) = \dfrac{\Gamma^{(1)}\Gamma^{(2)}}{\left(\frac{\Gamma^{(1)}+\Gamma^{(2)}}{2}\right)^2} \leq \underline{1}$ (equal only if $\Gamma^{(1)} = \Gamma^{(2)}$)

c) $$\boxed{\langle \hat{N} \rangle = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{f_1(E)\Gamma^{(1)}(E) + f_2(E)\Gamma^{(2)}(E)}{(E-\varepsilon)^2 + [\bar{\Gamma}(E)]^2}}$$

Now suppose $\Gamma^{(1)} = \text{const.}$, $\Gamma^{(2)} = \text{const.}$ and $T_\alpha \to 0$

$\Rightarrow$

$$\langle \hat{N} \rangle = \int_{-\infty}^{\mu_1} \frac{dE}{2\pi} \frac{\Gamma^{(1)}}{(E-\varepsilon)^2 + \bar{\Gamma}^2}$$

$$+ \int_{-\infty}^{\mu_2} \frac{dE}{2\pi} \frac{\Gamma^{(2)}}{(E-\varepsilon)^2 + \bar{\Gamma}^2}$$

Let $x = \frac{E-\varepsilon}{\bar{\Gamma}}$, $dE = \bar{\Gamma}\, dx$

$$\boxed{\begin{aligned}\langle \hat{N} \rangle = &\frac{\Gamma^{(1)}}{\Gamma^{(1)} + \Gamma^{(2)}} \left(\frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{\mu_1 - \varepsilon}{\bar{\Gamma}} \right) \right) \\ &+ \frac{\Gamma^{(2)}}{\Gamma^{(1)} + \Gamma^{(2)}} \left(\frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{\mu_2 - \varepsilon}{\bar{\Gamma}} \right) \right)\end{aligned}}$$

Note: $0 \leq \langle \hat{N} \rangle \leq 1$