Physics 560A

Extra Credit Problems Solutions

1)

$$\vec{x}_\ell = \frac{1}{\sqrt{N}} \sum_{\vec{k},\nu} \vec{\epsilon}_{\vec{k}\nu}\, Q_{\vec{k}\nu}\, e^{i\vec{k}\cdot\vec{R}_\ell}$$

$$Q_{\vec{k}\nu} = \sqrt{\frac{\hbar}{2m\omega_{\vec{k}\nu}}} \left(a_{\vec{k}\nu} + a^{+}_{-\vec{k}\nu}\right)$$

$$\langle \vec{x}_\ell^{\,2} \rangle = \frac{\hbar}{2mN} \sum_{\substack{\vec{k},\vec{k}' \\ \nu,\nu'}} \vec{\epsilon}_{\vec{k}\nu}\cdot\vec{\epsilon}_{\vec{k}'\nu'}\, e^{i(\vec{k}+\vec{k}')\cdot\vec{R}_\ell} \Big/ \sqrt{\omega_{\vec{k}\nu}\omega_{\vec{k}'\nu'}} \times \underbrace{\left\langle \left(a_{\vec{k}\nu} + a^{+}_{-\vec{k}\nu}\right)\left(a_{\vec{k}'\nu'} + a^{+}_{-\vec{k}'\nu'}\right)\right\rangle}_{\delta_{-\vec{k},\vec{k}'}\,\delta_{\nu\nu'}\left(1 + 2\langle n_{\vec{k}\nu}\rangle\right)}$$

$$\langle \vec{x}_\ell^{\,2} \rangle = \frac{\hbar}{2mN} \sum_{\vec{k},\nu} \frac{2\langle n_{\vec{k}\nu}\rangle + 1}{\omega_{\vec{k}\nu}}$$

since $\vec{\epsilon}_{\vec{k}\nu}\cdot\vec{\epsilon}_{\vec{k}\nu'} = \delta_{\nu\nu'}$

$$\langle n_{\vec{k}\nu} \rangle = \frac{1}{e^{\beta\hbar\omega_{\vec{k}\nu}} - 1}$$

Let's consider high-T limit first:

$$\langle n_{\vec{k}\nu} \rangle \approx \frac{1}{1 + \frac{\hbar\omega_{\vec{k}\nu}}{k_B T} + \cdots - 1} \simeq \frac{k_B T}{\hbar \omega_{\vec{k}\nu}}$$

$k_B T \gg \hbar\omega_{max}$:

$$\langle \vec{x}_\ell^{\,2} \rangle \simeq \frac{1}{N} \sum_{\vec{k},\nu} \frac{k_B T}{m\,\omega_{\vec{k}\nu}^2}$$

Back to general case, use 3D Debye model:

$$D(\omega) = \frac{3V\omega^2}{2\pi^2 v^3}, \quad \omega_D = v\left(6\pi^2 \frac{N}{V}\right)^{1/3}$$

$$\omega_{\vec{k}\nu} = v|\vec{k}|, \quad \nu = 1, 2, 3$$

$$\langle \vec{x}_\ell^{\,2} \rangle = \frac{\hbar}{2mN} \int_0^{\omega_D} \frac{D(\omega)\,(2n(\omega) + 1)}{\omega}\, d\omega$$

$$\langle \vec{x}_\ell^{\,2} \rangle = \frac{3V}{2\pi^2 N} \frac{\hbar}{m v^3} \int_0^{\omega_D} d\omega\, \omega \left(n(\omega) + \frac{1}{2} \right)$$

$$\lim_{T \to 0} \langle \vec{x}_\ell^{\,2} \rangle = \frac{3V}{8\pi^2 N} \frac{\hbar \omega_D^2}{m v^3} = \langle \vec{x}_\ell^{\,2} \rangle_0$$

$$\langle \vec{x}_\ell^{\,2} \rangle_0 = \frac{\hbar}{m\omega_D} \frac{9}{4} \quad \text{(obviously finite!)}$$

$$\langle \vec{x}_\ell^{\,2} \rangle = \langle \vec{x}_\ell^{\,2} \rangle_0 + \frac{9}{2} \frac{\hbar}{m\omega_D^3} \int_0^{\omega_D} d\omega \frac{\omega}{e^{\beta\hbar\omega} - 1}$$

i) <u>high-T limit:</u> $\frac{1}{e^{\beta\hbar\omega} - 1} \approx \frac{k_B T}{\hbar\omega}$

$$\langle \vec{x}_\ell^{\,2} \rangle \cong \langle \vec{x}_\ell^{\,2} \rangle_0 + \frac{9}{2} \frac{\hbar}{m\omega_D^3} \frac{k_B T}{\hbar} \int_0^{\omega_D} d\omega$$

$$\langle \vec{x}_\ell^{\,2} \rangle \simeq \langle \vec{x}_\ell^{\,2} \rangle_0 + \frac{9}{2} \frac{k_B T}{m\omega_D^2}, \quad k_B T \gg \hbar\omega_D$$

$\rightarrow$ finite

(ii) low-T limit: let $y = \beta\hbar\omega$

$$\langle \vec{x}_\ell^{\,2} \rangle = \langle \vec{x}_\ell^{\,2} \rangle_0 + \frac{9}{2}\frac{\hbar}{m\omega_D^3}\left(\frac{k_B T}{\hbar}\right)^2 \underbrace{\int_0^{y_D \to \infty} \frac{y\,dy}{e^y - 1}}_{\pi^2/6}$$

$$\langle \vec{x}_\ell^{\,2} \rangle = \langle \vec{x}_\ell^{\,2} \rangle_0 + \frac{3\pi^2}{4}\frac{k_B^2 T^2}{m\hbar\omega_D^3} \longrightarrow \text{finite}$$

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2) Quantum dot (fermions)

Consider basis states for the q-dot Hilbert space:

$$|0\rangle, \quad |\uparrow\rangle = d_\uparrow^\dagger |0\rangle$$

$$|\downarrow\rangle = d_\downarrow^\dagger |0\rangle, \quad |\uparrow\downarrow\rangle = d_\uparrow^\dagger d_\downarrow^\dagger |0\rangle$$

These are eigenstates of $\hat{N}$ with eigenvalues

$$\hat{N}|0\rangle = 0 \qquad \hat{N}|\downarrow\rangle = |\downarrow\rangle$$

$$\hat{N}|\uparrow\rangle = |\uparrow\rangle \qquad \hat{N}|\uparrow\downarrow\rangle = 2|\uparrow\downarrow\rangle$$

Thus these are the energy eigenstates, with eigenvalues

$$E_0 = 0 \qquad E_\uparrow = \varepsilon + \frac{U}{2}$$

$$E_\downarrow = \varepsilon + \frac{U}{2} \qquad E_{\uparrow\downarrow} = 2\varepsilon + 2U$$

<u>Coulomb blockade</u> when q-dot is in contact with reservoir at chemical pot. μ, 1st electron is added when

$$\mu_1 = E_\uparrow - E_0 = \varepsilon + \frac{U}{2}$$

2nd electron is added when

$$\mu_2 = E_{\uparrow\downarrow} - E_\uparrow = \varepsilon + \frac{3U}{2}$$

$$\mu_2 - \mu_1 = U$$