Physics 560A HW 4
Solutions

1) a) The eigenstates of $\hat{N}$ form a complete basis for Hilbert space. Thus, it suffices to see how a particular operator acts on a general eigenstate of $\hat{N}$:

$$\hat{N} |\psi_N\rangle = N |\psi_N\rangle.$$

Such a state $|\psi_N\rangle$ can be written

$$|\psi_N\rangle = \prod_j \frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}} |0\rangle,$$

where $N = \sum_j n_j$.

Consider the product

$\hat{A} = a_i a_j^\dagger a_k^\dagger a_\ell \cdots$, where there are a total of n_+ creation operators and n_- annihilation operators in the product. From the form of $|\Psi\rangle$, it is apparent that

$$\hat{N}\,\hat{A}\,|\Psi_N\rangle = (N + n_+ - n_-)\,\hat{A}\,|\Psi_N\rangle$$

Thus

$$[\hat{N}, \hat{A}]|\Psi_N\rangle = (N + n_+ - n_- - N)\,\hat{A}\,|\Psi_N\rangle$$

$$= (n_+ - n_-)\,\hat{A}\,|\Psi_N\rangle$$

$$\Rightarrow \quad [\hat{N}, \hat{A}] = 0 \quad \text{iff} \quad n_+ = n_- .$$

b) For fermions, let

$$\hat{N} = \sum_\nu c_\nu^\dagger a_\nu$$

Consider the product

$$\hat{B} = c_i c_j^\dagger c_k^\dagger c_\ell \cdots$$

where, again, there are a total of n_+ creation operators and n_- annihilation operators.

Consider also an eigenstate of $\hat{N}$:

$\hat{N}|\Psi_N\rangle = N|\Psi_N\rangle$. $|\Psi_N\rangle$ has the form:

$$|\Psi_N\rangle = \prod_{j=j_1}^{j_N} c_j^\dagger |0\rangle .$$

These states (cf. Slater determinants) form a complete basis for the many-fermion Hilbert space, so it suffices to see how a operator acts on such a state $|\Psi_N\rangle$.

There are cases where $\hat{B} = 0$, for example

$$\hat{B} = c_j^{+} c_j^{+} = 0.$$

These cases must be excluded from the argument, since every operator "commutes" with 0. Otherwise, we have

$$\hat{N}\hat{B}|\psi_N\rangle = (N + n_+ - n_-)\hat{B}|\psi_N\rangle$$

and

$$[\hat{N}, \hat{B}]|\psi_N\rangle = (n_+ - n_-)\hat{B}|\psi_N\rangle.$$

Since this is true for all basis states, it follows that

$$[\hat{N}, \hat{B}] = 0 \quad \text{iff} \quad n_+ = n_-$$

2) $H^{(2)} = \frac{1}{2} \sum_{ijlm} V_{ijlm}\, c_j^+ c_m^+ c_l c_i$

(sorry about the typo!)

$$|\mu\nu\rangle = c_\mu^+ c_\nu^+ |0\rangle$$

$$c_l c_i c_\mu^+ c_\nu^+ |0\rangle = c_l (\delta_{i\mu} - c_\mu^+ c_i) c_\nu^+ |0\rangle$$

$$= (\delta_{i\mu} c_l c_\nu^+ - c_l c_\mu^+ c_i c_\nu^+)|0\rangle$$

$$= \delta_{i\mu}(\delta_{l\nu} - c_\nu^+ c_l)|0\rangle$$
$$- c_l c_\mu^+ (\delta_{i\nu} - c_\nu^+ c_i)|0\rangle$$

$$= \delta_{i\mu}\delta_{l\nu}|0\rangle - \delta_{i\nu}\delta_{l\mu}|0\rangle$$

$$\langle\mu\nu| c_j^+ c_m^+ = \langle 0| c_\nu c_\mu c_j^+ c_m^+$$

$$= (c_m c_j c_\mu^+ c_\nu^+ |0\rangle)^+$$

$$= ((\delta_{j\mu}\delta_{m\nu} - \delta_{j\nu}\delta_{m\mu})|0\rangle)^+$$

$$\langle \mu\nu | H^{(2)} | \mu\nu \rangle$$

$$= \frac{1}{2} \sum_{ijlm} V_{ijlm} \left(\delta_{j\mu} \delta_{m\nu} - \delta_{j\nu} \delta_{m\mu} \right) \left(\delta_{i\mu} \delta_{l\nu} - \delta_{i\nu} \delta_{l\mu} \right)$$

$$= \frac{1}{2} \left(V_{\mu\mu\nu\nu} + V_{\nu\nu\mu\mu} - V_{\mu\nu\nu\mu} - V_{\nu\mu\mu\nu} \right)$$

$$= V_{\mu\mu\nu\nu} - V_{\mu\nu\nu\mu} \qquad \text{Q.E.D.}$$

Physics 560 Solutions

Problem set # 4 (cont.)

3)

a) The total # of states with energy $\leq \mathcal{E}$ is given by the area of a disk in phase space

$$N(\mathcal{E}) = 2\int_{|k| \leq \sqrt{\frac{2m\mathcal{E}}{\hbar^2}}} \left(\frac{L}{2\pi}\right)^2 d^2k ,$$

where the prefactor is for spin.

$$N(\mathcal{E}) = \frac{A}{2\pi^2}\,\pi\,\frac{2m\mathcal{E}}{\hbar^2} = \frac{mA\mathcal{E}}{\pi\hbar^2}$$

$$D(\mathcal{E}) = \frac{dN}{d\mathcal{E}} = \frac{mA}{\pi\hbar^2}$$

At zero temperature,
The total # of particles
is

$$N = 2 \int_{|k| \le k_F} \left(\frac{L}{2\pi}\right)^2 d^2k$$

$$= \frac{A}{2\pi^2} \pi k_F^2 = \frac{A k_F^2}{2\pi}$$

$$= \frac{mA}{\hbar^2 \pi} \varepsilon_F \qquad \left(\varepsilon_F = \frac{\pi \hbar^2}{m} \frac{N}{A} \right)$$

Thus $D(\varepsilon) = \dfrac{N}{\varepsilon_F}$

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b) $N = \int_0^\infty d\varepsilon \, D(\varepsilon) f(\varepsilon)$

$$\frac{N}{A} = \frac{m}{\pi \hbar^2} \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$

$$\varepsilon_F = \frac{\pi \hbar^2}{m} \frac{N}{A} = \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$

Let $x = \beta(\varepsilon - \mu)$

$$\varepsilon_F = \frac{1}{\beta}\int_{-\beta\mu}^{\infty} \frac{dx}{e^x + 1}$$

$$= \frac{1}{\beta}\int_{-\beta\mu}^{\infty} \frac{dx\ e^{-x}}{1 + e^{-x}}$$

$$= -\frac{1}{\beta} \ln\left(1 + e^{-x}\right)\Big|_{-\beta\mu}^{\infty}$$

$$= \frac{1}{\beta}\ln\left(1 + e^{\beta\mu}\right)$$

$$\beta\varepsilon_F = \ln\left(1 + e^{\beta\mu}\right)$$

$$e^{\beta\varepsilon_F} = 1 + e^{\beta\mu}, \quad \beta\mu = \ln\left(e^{\beta\varepsilon_F} - 1\right)$$

$$\boxed{\mu = \beta^{-1}\ln\left(e^{\beta\varepsilon_F} - 1\right)}$$

Note that $\lim_{T\to 0} \mu(T) = \varepsilon_F$, as it should.

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c) $$E = \int_0^\infty d\varepsilon \frac{D(\varepsilon)\,\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= \frac{N}{\varepsilon_F} \int_0^\infty d\varepsilon \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$C_V = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta}\frac{\partial \beta}{\partial T} = -\frac{1}{k_B T^2}\frac{\partial E}{\partial \beta}$$

$$= \frac{N}{\varepsilon_F}\frac{1}{k_B T^2}\int_0^\infty d\varepsilon \frac{\varepsilon(\varepsilon-\mu)\,e^{\beta(\varepsilon-\mu)}}{\left(e^{\beta(\varepsilon-\mu)}+1\right)^2}$$

$$= \frac{N}{4\varepsilon_F}\frac{1}{k_B T^2}\int_0^\infty d\varepsilon \frac{\varepsilon(\varepsilon-\mu)}{\cosh^2\left[\beta\frac{(\varepsilon-\mu)}{2}\right]}$$

Let $\beta\frac{(\varepsilon-\mu)}{2} = x$

$$C_V = \frac{N}{4\varepsilon_F}\frac{1}{k_B T^2}\left(\frac{2}{\beta}\right)^3 \int_{-\frac{\beta\mu}{2}}^{\infty} dx \, \frac{x^2 + \frac{\beta\mu}{2}x}{\cosh^2 x}$$

$$C_V \underset{T \ll T_F}{\simeq} \frac{2N}{\varepsilon_F} k_B^2 T \underbrace{\int_{-\infty}^{\infty} dx \, \frac{x^2 + \cancel{\frac{\beta\mu}{2}x}^{\,\to 0}}{\cosh^2 x}}$$

$$= 2Nk_B \frac{k_B T}{\varepsilon_F} \times \frac{\pi^2}{6}$$

$$\boxed{C_V = \frac{\pi^2}{3} N k_B \frac{k_B T}{\varepsilon_F}}$$