Lecture 23
Introduction to Nonequilibrium
Green's functions

We have seen that a general (time-independent) observable one-body can be written in 2nd quantization as

\[ \hat{\mathcal{A}} = \sum_{n,m} Q_{nm} \hat{a}^+_n \hat{a}_m, \]

where the sum runs over a complete single-particle basis. The most general time-dependent one-body observable has the form
\[ \hat{A}(t_1, t_2) = \sum_{n,m} A_{nm}(t, t_2) \frac{\rho_n(t_1)}{\rho_m(t_2)} \]

Thus, the expectation values of all one-particle observables can be determined from the Green's functions.

\[ G_{nm}(t_1, t_2) = \langle d_m^+(t_2) d_n(t_1) \rangle \]

\[ G_{nm}(t_1, t_2) = -i \langle d_n(t_1) d_m^+(t_2) \rangle \]

Here \( \langle \rangle \) implies the statistical and QM average. How to compute \( G \) out of equilibrium?
Interaction rep.

\[ H = H_0 + V \]

\[ |\psi(t)\rangle = e^{iH_0 t} |\psi(0)\rangle \]

\[ \hat{O}(t) = e^{iH_0 t} \hat{O} e^{-iH_0 t} \]

Note \([H_0, V] \neq 0\), so

\[ e^{iH_0 t} e^{-iH_0 t} \neq e^{iV t} \]

\[ \langle \psi(t) | \hat{O}(t) | \psi(t) \rangle = \langle \psi(0) | e^{iH_0 t} \hat{O} e^{-iH_0 t} \psi(0) \rangle \]

Same as in Sch. rep. / Heisenberg rep.
Propagator

\[ U(t) = e^{iH_0 t - i\mathcal{H}t} \]

\[ |\psi(t)\rangle = U(t) |\psi(0)\rangle \]

\[ \frac{2}{\mathcal{H}} |\psi(t)\rangle = i e^{iH_0 t} (H_0 - \mathcal{H})^{-i\mathcal{H}t} |\psi(0)\rangle \]

\[ \begin{aligned} & = -i \hat{\mathcal{V}}(t) |\psi(t)\rangle \\ \text{time dep. of states deformed by } \hat{\mathcal{V}}(t) \quad \text{time dep. of operators deformed by } H_0. \end{aligned} \]

\[ U(0) = 1 \]
\[ \frac{\partial U(t)}{\partial t} = i e^{i H_0 t} (H_0 - H) e^{-i H t} \]
\[ = -i e^{i H_0 t} V (e^{-i H_0 t} e^{-i H t}) e^{-i H t} \]
\[ = -i \hat{V}(+) U(t) \]

\[ U(t) = U(0) - i \int_0^t dt' \hat{V}(+) U(t') \]

The solution is

\[ U(t) = T \exp \left[ -i \int_0^t dt' \hat{V}(+) \right] \]

\[ S\text{-matrix} \]

\[ \left| \Psi(t) \right> = S(t, +r) \left| \Psi(+) \right> \]
\[ \mathcal{S}(t,t') = U(t) U^+(t') \]

**Proof:**

\[ |\psi(t)\rangle = U(t) |\psi_0\rangle = \mathcal{S}(t,t') U(t') |\psi_0\rangle \]

Moreover,

\[ \mathcal{S}(t,t') = T \left[ \exp \left( -i \int_{t'}^{t} dt'' \tilde{V}(t'') \right) \right] \]

The crux of the matter

\[ |\psi(t)\rangle = \mathcal{S}(t,-\infty) |\psi(-\infty)\rangle \]