Physics 560A Midterm

Solutions

1) $$E = \sum_{k \in BZ} \hbar\omega(k)\left(\langle n_k\rangle + \frac{1}{2}\right)$$

$$\omega(k) = \sqrt{\tilde{C}(k)/m} \qquad \langle n_k\rangle = \frac{1}{e^{\beta\hbar\omega(k)} - 1}$$

$$\sum_k \longrightarrow \frac{L}{2\pi}\int_{-\pi/a}^{\pi/a} dk$$

let $V_s = \lim\limits_{k\to 0} \frac{d\omega}{dk}$

$$C_V = \frac{\partial E}{\partial T}\Big|_V = \frac{L}{2\pi}\int_{-\pi/a}^{\pi/a} dk \quad \hbar\omega(k) \quad \frac{\partial\langle n_k\rangle}{\partial T}$$

$$\frac{\partial\langle n_k\rangle}{\partial T} = -\frac{\hbar\omega(k)\, e^{\beta\hbar\omega(k)}}{\left(e^{\beta\hbar\omega} - 1\right)^2} \cdot -\frac{1}{k_B T^2}$$

$$C_V = \frac{L}{2\pi k_B T^2} \int_{-\pi/a}^{\pi/a} dk \, \frac{(\hbar\omega(k))^2 \, e^{\beta\hbar\omega(k)}}{\left(e^{\beta\hbar\omega} - 1\right)^2}$$

For low temperatures, integrand is exponentially small except for small k. Expand: $\omega(k) \approx v_s|k|$

Let $\beta\hbar v_s|k| = x$

$$C_V \simeq \frac{L}{\pi k_B T^2} \int_0^{\beta\hbar v_s \pi/a} dx \, \frac{x^2 e^x}{(e^x - 1)^2} \, \frac{1}{\hbar v_s \beta^3}$$

$$\frac{C_V}{L k_B} \simeq \frac{k_B T}{\pi\hbar v_s} \underbrace{\int_0^{\infty} dx \, \frac{x^2 e^x}{(e^x - 1)^2}}_{I = \pi^2/3}$$

$$\boxed{\frac{C_V}{L k_B} = \frac{\pi}{3} \frac{k_B T}{\hbar v_s}}$$

Low T means

$\beta \frac{\hbar v_s}{a} \ll 1$

2) $$|\mu\rangle = c_\mu^\dagger |0\rangle$$

$$\langle \mu | H^{(2)} | \mu \rangle = \frac{1}{2}\int d^3x \int d^3y \, V(\vec{x}-\vec{y})$$
$$\times \langle \mu | \hat{\rho}(\vec{x}) \, \hat{\rho}(\vec{y}) | \mu \rangle$$

$$c_\mu^\dagger = \int d^3x' \, \psi_\mu(\vec{x}') \, \hat{\psi}^\dagger(\vec{x}')$$

$$\begin{aligned}
\hat{\rho}(\vec{x})\,\hat{\rho}(\vec{y}) &= \hat{\psi}^\dagger(\vec{x})\,\hat{\psi}(\vec{x})\,\hat{\psi}^\dagger(\vec{y})\,\hat{\psi}(\vec{y}) \\
&= \delta(\vec{x}-\vec{y})\,\hat{\psi}^\dagger(\vec{x})\,\hat{\psi}(\vec{y}) \pm \hat{\psi}^\dagger(\vec{x})\,\hat{\psi}^\dagger(\vec{y}) \\
&\qquad \times \hat{\psi}(\vec{x})\,\hat{\psi}(\vec{y}) \\
&= \delta(\vec{x}-\vec{y})\,\hat{\rho}(\vec{x}) + \hat{\psi}^\dagger(\vec{x})\,\hat{\psi}^\dagger(\vec{y})\,\hat{\psi}(\vec{y})\,\hat{\psi}(\vec{x})
\end{aligned}$$

$$\begin{aligned}
\langle \mu | \hat{\rho}(\vec{x})\,\hat{\rho}(\vec{y}) | \mu \rangle &= \delta(\vec{x}-\vec{y}) \langle \mu | \hat{\rho}(\vec{x}) | \mu \rangle \\
&= \delta(\vec{x}-\vec{y})\,|\psi_\mu(\vec{x})|^2
\end{aligned}$$

$$\langle \mu | H^{(2)} | \mu \rangle = \frac{1}{2}\int d^3x \int d^3y \, \delta(\vec{x}-\vec{y})\, V(\vec{x}-\vec{y})\, |\psi_\mu(\vec{x})|^2$$

$$\langle \mu | H^{(2)} | \mu \rangle = \frac{1}{2} V(0) \int d^3x \, |\psi_\mu(\vec{x})|^2$$

$$= \frac{1}{2} V(0)$$

$\longrightarrow$ self-interaction