Physics 560A Midterm solutions

$$\boxed{\frac{C_V}{N k_B} = \frac{\pi^2}{2}\frac{T}{T_F} + \frac{12\pi^4}{5}\left(\frac{T}{\theta_D}\right)^3}$$

Derivation:

$$C_V = C_V\big|_{el} + C_V\big|_{ph}$$

i) <u>phonons</u>

$$E = \underbrace{U_0 + \sum_{\vec{k},s} \frac{\hbar\omega_s(\vec{k})}{2}}_{E_0} + \sum_{\vec{k},s} \frac{\hbar\omega_s(\vec{k})}{e^{\beta\hbar\omega_s(\vec{k})} - 1}$$

$$\sum_{\vec{k},s} \to \sum_s V\int \frac{d^3k}{(2\pi)^3} \longrightarrow \int_0^{\omega_D} d\omega\, D(\omega)$$

Debye model: $\omega_s(\vec{k}) = v|\vec{k}|, \quad s = 1,2,3$

$$D(\omega) = \frac{3V\omega^2}{2\pi^2 v^3}, \qquad \omega_D = v\left(\frac{6\pi^2 N}{V}\right)^{1/3}$$

N = # of unit cells in volume V

$$E - E_0 = \int_0^{\omega_D} \frac{D(\omega)\,\hbar\omega}{e^{\beta\hbar\omega} - 1}\, d\omega$$

$$C_V\Big|_{ph} = \frac{\partial E}{\partial T}\Big|_V = \frac{3V\hbar^2}{2\pi^2 v^3 k_B T^2} \int_0^{\omega_D} \frac{\omega^4 e^{\beta\hbar\omega}\, d\omega}{\left(e^{\beta\hbar\omega} - 1\right)^2}$$

let $x = \beta\hbar\omega$, $\theta_D = \hbar\omega_D / k_B$

$$C_V\Big|_{ph} = 9Nk_B\left(\frac{T}{\theta_D}\right)^3 \int_0^{\theta_D/T} \frac{x^4 e^x\, dx}{\left(e^x - 1\right)^2}$$

For $T \ll \theta_D$, can take upper limit

$\frac{\theta_D}{T} \to \infty$:

$$C_V\Big|_{ph} = 9Nk_B\left(\frac{T}{\theta_D}\right)^3 \underbrace{\int_0^{\infty} \frac{x^4 e^x\, dx}{\left(e^x - 1\right)^2}}_{\frac{4\pi^4}{15}}$$

$$\boxed{C_V\Big|_{ph} = \frac{12\pi^4}{5} N k_B \left(\frac{T}{\theta_D}\right)^3}$$

ii) Electrons

$$E = \int_{-\infty}^{\infty} d\varepsilon \ \varepsilon D(\varepsilon) f(\varepsilon)$$

Sommerfeld expansion:

$$E \simeq \int_{-\infty}^{\mu} d\varepsilon \ \varepsilon D(\varepsilon) + \frac{\pi^2}{6}(k_B T)^2 \frac{d}{d\mu}(\mu D(\mu))$$

$$= \overbrace{\int_{-\infty}^{\varepsilon_F} d\varepsilon \ \varepsilon D(\varepsilon)}^{E_0} + \int_{\varepsilon_F}^{\mu} d\varepsilon \ \varepsilon D(\varepsilon)$$

$$+ \frac{\pi^2}{6}(k_B T)^2 \left[D(\mu) + \mu D'(\mu)\right]$$

$$\simeq E_0 + (\mu - \varepsilon_F)\varepsilon_F D(\varepsilon_F)$$

$$+ \frac{\pi^2}{6}(k_B T)^2 \left[D(\varepsilon_F) + \varepsilon_F D'(\varepsilon_F)\right]$$

But $\mu \simeq \varepsilon_F - \frac{\pi^2}{6}(k_B T)^2 \frac{D'(\varepsilon_F)}{D(\varepsilon_F)}$

$$\Rightarrow \quad E - E_0 \simeq \frac{\pi^2}{6}(k_B T)^2 D(\varepsilon_F)$$

$$C_v\big|_{el} = \frac{\pi^2}{3} k_B^2 \, T \, D(\varepsilon_F)$$

$$D(\varepsilon_F) = \frac{3}{2}\frac{N}{\varepsilon_F} \quad (3D)$$

$$\boxed{C_v\big|_{el} = \frac{\pi^2}{2} N k_B \frac{k_B T}{\varepsilon_F}}$$