Exercises for Physics 560A

Problem Set 4; Due Friday, October 1

1) Verify that the total number operator

$$\hat{N} = \sum_\nu a_\nu^\dagger a_\nu$$

for a system of bosons commutes with a product of creation and annihilation operators if and only if the number of $a^\dagger$'s equals the number of a's in the product.

b) Repeat the derivation for fermions.

2) Two-body interaction

Verify that the expectation value of the 2-fermion interaction

$$H^{(2)} = \frac{1}{2} \sum_{ij\ell m} V_{ij\ell m} c_j^\dagger c_m^\dagger c_\ell c_i$$

in the state $|\mu\nu\rangle = c_\mu^\dagger c_\nu^\dagger |0\rangle$ is

$$\langle \mu\nu | H^{(2)} | \mu\nu \rangle = V_{\mu\mu\nu\nu} - V_{\mu\nu\nu\mu}.$$

Here use has been made of the symmetry $V_{ij\ell m} = V_{\ell m ij}$.

3) Two-dimensional electron gas

Consider a system of N non-interacting electrons in a two-dimensional box in the form of a square of side L, with area $A = L^2$. For simplicity, you may impose periodic boundary conditions.

a) Show that the density of states is independent of energy, and is given by

$$D(E) = \frac{mA}{\pi\hbar^2} = \frac{N}{E_F},$$

where E_F is the energy of the highest occupied state at $T = 0$.

b) Show that the chemical potential is given by

$$\mu = \beta^{-1} \ln\left(e^{\beta E_F} - 1\right),$$

where $\beta = 1/k_B T$.

c) Show that for temperatures $T \ll E_F/k_B$, the specific heat is

$$C_V \simeq \frac{\pi^2}{3} N k_B \frac{k_B T}{E_F}.$$