Exercises for Physics 560A

Problem Set 3; Due Friday, September 23

1) Phonons in 3D

Verify that if the Fourier variables for phonons in a 3D Bravais lattice are defined by

\[ Q_{k\nu} = \epsilon_{k\nu} \cdot N^{-1/2} \sum_{n=1}^{N} e^{-i \mathbf{k} \cdot \mathbf{R}_n} x_\ell, \]

\[ P_{k\nu} = \epsilon_{k\nu} \cdot N^{-1/2} \sum_{n=1}^{N} e^{i \mathbf{k} \cdot \mathbf{R}_n} p_\ell, \]

then the displacement and momentum operators for the \( \ell \)th atom are

\[ x_\ell = N^{-1/2} \sum_{k\nu} \epsilon_{k\nu} Q_{k\nu} e^{i \mathbf{k} \cdot \mathbf{R}_\ell}, \]

\[ p_\ell = N^{-1/2} \sum_{k\nu} \epsilon_{k\nu} P_{k\nu} e^{-i \mathbf{k} \cdot \mathbf{R}_\ell}. \]

b) The corresponding creation and annihilation operators are

\[ a_{k\nu} = \sqrt{\frac{\hbar \omega_{k\nu}}{2m}} Q_{k\nu} + \frac{iP_{-k\nu}}{\sqrt{2m\hbar \omega_{k\nu}}}, \]

\[ a_{k\nu}^\dagger = \sqrt{\frac{\hbar \omega_{k\nu}}{2m}} Q_{-k\nu} - \frac{iP_{k\nu}}{\sqrt{2m\hbar \omega_{k\nu}}}. \]

Show that

\[ [a_{k\nu}, a_{k'\nu}^\dagger] = \delta_{kk'} \delta_{\nu \nu'}. \]

2) Two-dimensional Debye model

Repeat the steps in the derivation of the Debye model for a two-dimensional crystal, assuming there are only two acoustic modes (one longitudinal and one transverse in-plane) with frequencies

\[ \omega_s(\mathbf{k}) = v|\mathbf{k}|, \quad s = 1, 2, \]

where \( v \) is the speed of sound.

a) Determine the Debye frequency and the phonon density of states.
b) Write down a general expression for the thermal average energy of the system.

c) Show that the specific heat obeys a $T^2$ law in two dimensions.

3) Thermal fluctuations of a 2D crystal

Using the formalism from problem 1, calculate the mean-square displacement of an atom in the two-dimensional crystal of problem 2. Show that $\langle x_i^2 \rangle \to \infty$ for $T > 0$. 