

# Energy-time uncertainty

Consider a quantum system with

$\hat{H}$  independent of  $t$ .

$$\text{Let } G^r(t) = -\frac{i}{\hbar} \theta(t) \hat{U}(t)$$

$$= -\frac{i}{\hbar} \theta(t) \exp\left(-i \frac{\hat{H} t}{\hbar}\right)$$

= retarded Green's function

$$i\hbar \frac{d}{dt} G^r(t) = \delta(t) + \hat{H} G^r(t)$$

$$\left(i\hbar \frac{d}{dt} - \hat{H}\right) G^r(t) = \delta(t) \mathbb{1}$$

An open quantum system can  
be described by adding a non-Hermitian  
self-energy  $\hat{\Sigma}^r$  to  $\hat{H}$ :

$$\hat{H} \rightarrow \tilde{H} = \hat{H} + \hat{\Sigma}^r$$

Suppose, for simplicity

$$\hat{\Sigma}^r = -\frac{i}{2} \sum_n \Gamma_n |n\rangle\langle n|,$$

where  $\hat{H}|n\rangle = E_n|n\rangle$ .

$G^r$  now satisfies

$$\left( i\hbar \frac{d}{dt} - \hat{H} - \hat{\Sigma}^r \right) G^r(t) = \delta(t) \mathbb{1}$$

Define  $G^r(\omega) = \int_{-\infty}^{\infty} dt G^r(t) e^{i\omega t}$

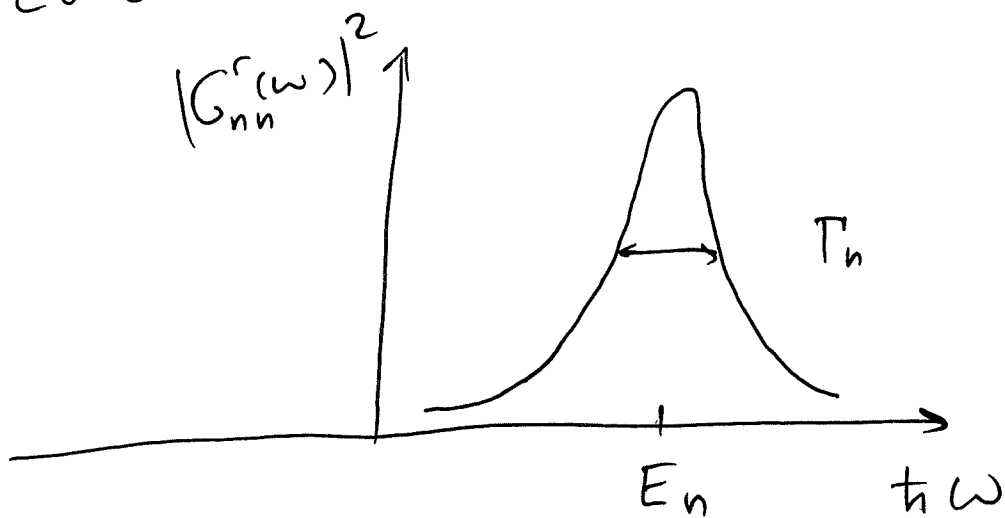
$$\left( \hbar\omega \mathbb{1} - \hat{H} - \hat{\Sigma}^r \right) G^r(\omega) = \mathbb{1}$$

Solution:

$$G^r(\omega) = \sum_n \frac{|n\rangle\langle n|}{\hbar\omega - E_n + i\Gamma_n/2}$$

$$|G_{nn}^r(\omega)|^2 = \frac{1}{(\hbar\omega - E_n)^2 + (\Gamma_n/2)^2}$$

Lorentzian



$$\frac{\Gamma_n}{2} = \text{HWHM} = \Delta E \quad \text{uncertainty in energy}$$

Spectral function

$$A(\omega) = \frac{i}{2\pi} [G^r(\omega) - G^a(\omega)]$$

$$= \frac{1}{2\pi} \sum_n \frac{\Gamma_n |n\rangle \langle n|}{(\hbar\omega - E_n)^2 + (\Gamma_n/2)^2}$$

where  $G^a(\omega) = (G^r(\omega))^{\dagger}$ .

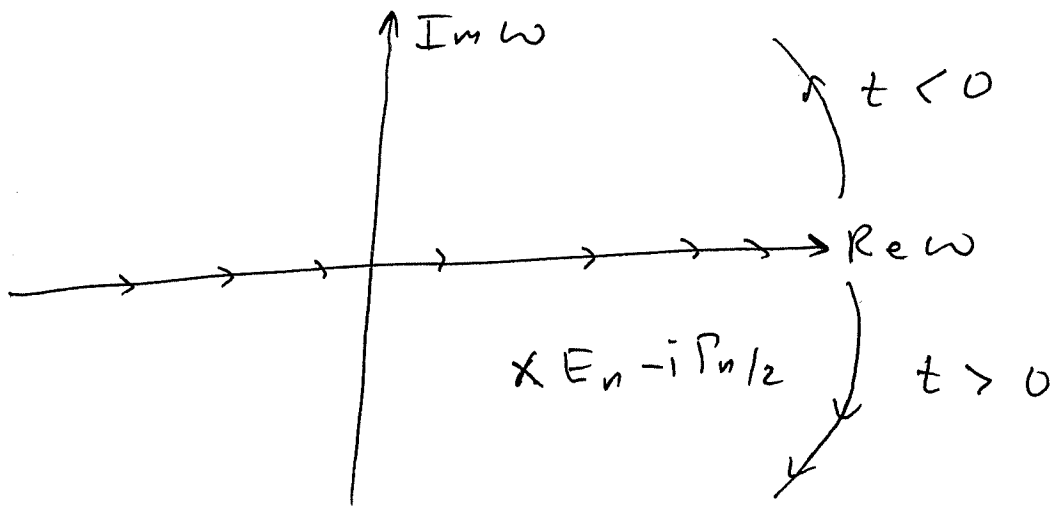
# Spectrum (DOS)

$$g(\omega) = \text{Tr} \{ A(\omega) \} = \frac{1}{2\pi} \sum_n \frac{\Gamma_n}{(\hbar\omega - E_n)^2 + (\Gamma_n/2)^2}$$

Fourier transforming back to the  $t$ -domain:

$$G^r(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^r(\omega) e^{-i\omega t}$$

$$= \sum_n |n\rangle\langle n| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\hbar\omega - E_n + i\Gamma_n/2}$$



$$G^r(t) = -\frac{i}{\hbar} \sum_n |n\rangle\langle n| e^{-\frac{iE_n t}{\hbar} - \frac{\Gamma_n}{2\hbar} t}$$

$$|C_{nn}^r(t)|^2 = \frac{1}{\hbar^2} e^{-\frac{\Gamma_n t}{\hbar}} \equiv \frac{1}{\hbar^2} e^{-t/\tau}$$

$$\text{lifetime } \tau = \frac{\hbar}{\Gamma_n} = \Delta t$$

$$\Delta E \Delta t = \frac{\Gamma_n}{2} \frac{\hbar}{\Gamma_n} = \frac{\hbar}{2}$$

energy-time uncertainty relation

Note:

$$\begin{aligned} C_{nn}^r(t) &= \langle n | \hat{U}(t) | n \rangle \left( -\frac{i}{\hbar} \theta(t) \right) \\ &= -\frac{i}{\hbar} \theta(t) \langle n(0) | n(t) \rangle \end{aligned}$$

cf. Eq. 2-1-66 Sakurai:

$$\langle n(0) | n(t) \rangle = C(t) = \text{"correlation amplitude"}$$