Lec. 15

Energy - time uncertainty

Consider a quantum system with $\hat{H}$ independent of t.

Let
$$G^r(t) = -\frac{i}{\hbar}\theta(t)\,\hat{U}(t)$$
$$= -\frac{i}{\hbar}\theta(t)\exp\left(-\frac{i\hat{H}t}{\hbar}\right)$$
= retarded Green's function

$$i\hbar\frac{d}{dt}G^r(t) = \delta(t) + \hat{H}\,G^r(t)$$

$$\left(i\hbar\frac{d}{dt} - \hat{H}\right)G^r(t) = \delta(t)\,\mathbb{1}$$

An <u>open quantum system</u> can be described by adding a non-Hermitian self-energy $\hat{\Sigma}^r$ to $\hat{H}$:

$$\hat{H} \rightarrow \tilde{H} = \hat{H} + \hat{\Sigma}^r$$

Suppose, for simplicity

$$\hat{\Sigma}^r = -\frac{i}{2} \sum_n \Gamma_n \, |n\rangle\langle n| \,,$$

where $\hat{H}|n\rangle = E_n |n\rangle$.

G^r now satisfies

$$\left(i\hbar \frac{d}{dt} - \hat{H} - \hat{\Sigma}^r\right) G^r(t) = \delta(t)\, \mathbb{1}$$

Define $\quad G^r(\omega) = \int_{-\infty}^{\infty} dt \; G^r(t)\, e^{i\omega t}$

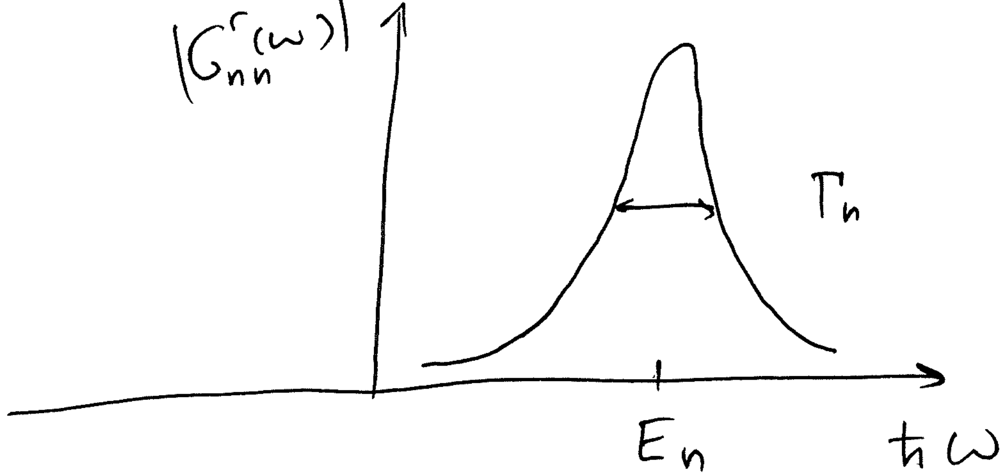
$$\left(\hbar\omega \mathbb{1} - \hat{H} - \hat{\Sigma}^r\right) G^r(\omega) = \mathbb{1}$$

Solution:

$$G^r(\omega) = \sum_n \frac{|n\rangle\langle n|}{\hbar\omega - E_n + i\Gamma_n/2}$$

$$|G^r_{nn}(\omega)|^2 = \frac{1}{(\hbar\omega - E_n)^2 + (\Gamma_n/2)^2}$$

Lorentzian



$\frac{\Gamma_n}{2}$ = HWHM = ΔE uncertainty in energy

Spectral function

$$A(\omega) = \frac{i}{2\pi}\left[G^r(\omega) - G^a(\omega)\right]$$

$$= \frac{1}{2\pi}\sum_n \frac{\Gamma_n |n\rangle\langle n|}{(\hbar\omega - E_n)^2 + (\Gamma_n/2)^2},$$

where $G^a(\omega) = (G^r(\omega))^\dagger$.

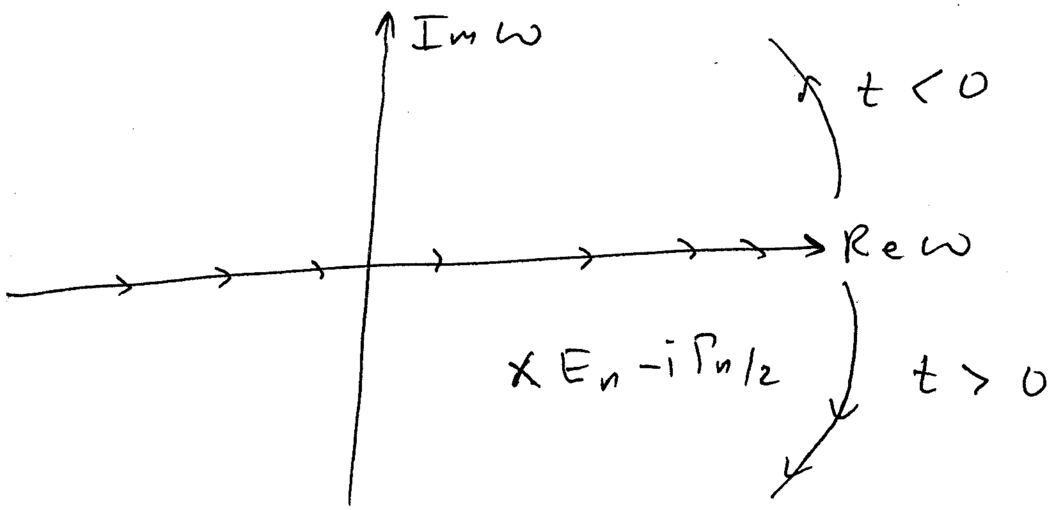
Spectrum (DOS)

$$g(\omega) = \mathrm{Tr}\{A(\omega)\} = \frac{1}{2\pi} \sum_n \frac{\Gamma_n}{(\hbar\omega - E_n)^2 + (\Gamma_n/2)^2}$$

Fourier transforming back to the t-domain:

$$G^r(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^r(\omega) e^{-i\omega t}$$

$$= \sum_n |n\rangle\langle n| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\hbar\omega - E_n + i\Gamma_n/2}$$



$$G^r(t) = -\frac{i}{\hbar}\theta(t) \sum_n |n\rangle\langle n| \, e^{-\frac{iE_n t}{\hbar} - \frac{\Gamma_n}{2\hbar} t}$$

$$|G^r_{nn}(t)|^2 = \theta(t)\frac{1}{\hbar^2} e^{-\frac{\Gamma_n t}{\hbar}} \equiv \frac{\theta(t)}{\hbar^2} e^{-t/\tau}$$

lifetime $\tau = \frac{\hbar}{\Gamma_n} = \Delta t$

$$\boxed{\Delta E \, \Delta t = \frac{\Gamma_n}{2} \frac{\hbar}{\Gamma_n} = \frac{\hbar}{2} \qquad \text{energy-time uncertainty relation}}$$

Note:

$$G^r_{nn}(t) = \langle n | \hat{U}(t) | n \rangle \left(-\frac{i}{\hbar}\theta(t)\right)$$

$$= -\frac{i}{\hbar}\theta(t) \langle n(0) | n(t) \rangle$$

cf. Eq. 2.1.66 Sakurai:

$\langle n(0) | n(t) \rangle = C(t) =$ "correlation amplitude"