Solutions

1) $\vec{A} = (-yB, 0, 0)$

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x - \frac{qB}{c}y\right)^2 + \frac{1}{2m}\hat{p}_y^2$$

$[\hat{H}, \hat{x}] = 0$, so let $\Psi(x,y) = \psi(y)e^{ikx}$

$$\hat{H}\Psi = \left[\frac{1}{2m}\left(\hbar k - \frac{qBy}{c}\right)^2 + \frac{1}{2m}\hat{p}_y^2\right]\psi(y)e^{ikx}$$

Dividing through by e^{ikx}, we have

$$E\psi(y) = \left[\frac{1}{2m}\hat{p}_y^2 + \frac{m\Omega^2}{2}(y-y_0)^2\right]\psi(y),$$

with $y_0 = \dfrac{\hbar c k}{qB}$, $\Omega = \dfrac{qB}{mc}$

a) $\boxed{E_n = \hbar\Omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \dots, \infty}$

b) $e^{ikL_x} = 1 \qquad k = \dfrac{2\pi n_x}{L_x}, \quad n \in \mathbb{Z}$

$0 \le y_0 \le L_y \qquad 0 \le \dfrac{\hbar c\, 2\pi n_x}{qBL_x} \le L_y$

$$0 \leq n_x \leq \frac{qB}{hc} L_x L_y$$

Degeneracy of each level is

$$\boxed{d_n = \frac{qB}{hc} L_x L_y}$$

2) a) $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$

Let $|\lambda\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, where

$\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$.

$$\hat{a}|\lambda\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{k=0}^{\infty} c_{k+1}\sqrt{k+1}\,|k\rangle$$

$$= \lambda|\lambda\rangle = \sum_{k=0}^{\infty} \lambda c_k |k\rangle$$

$$\Rightarrow \quad \lambda c_k = c_{k+1}\sqrt{k+1}$$

$$\frac{c_{k+1}}{c_k} = \frac{\lambda}{\sqrt{k+1}}$$

$$\frac{c_1}{c_0} = \lambda, \quad \frac{c_2}{c_1} = \frac{\lambda}{\sqrt{2}}, \quad \frac{c_3}{c_2} = \frac{\lambda}{\sqrt{3}}, \ldots$$

$$\frac{c_2}{c_0} = \frac{\lambda^2}{\sqrt{2!}}, \quad \frac{c_3}{c_0} = \frac{\lambda^3}{\sqrt{3!}}, \ldots$$

$$\frac{c_n}{c_0} = \frac{\lambda^n}{\sqrt{n!}}, \quad |\lambda\rangle = \sum_{n=0}^{\infty} c_0 \frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

$$1 = \langle\lambda|\lambda\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} = |c_0|^2 e^{\lambda^2}$$

$$C_0 = e^{-\lambda^2/2}. \qquad |\lambda\rangle = e^{-\lambda^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

Notice that $\quad e^{\lambda a^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{(\lambda a^\dagger)^n}{n!}|0\rangle$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle .$$

$$\therefore \quad |\lambda\rangle = e^{-\lambda^2/2} e^{\lambda a^\dagger} |0\rangle .$$

b) Suppose $\exists \; |\nu\rangle$ s.t. $a^\dagger |\nu\rangle = \nu |\nu\rangle$

Let $|\nu\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$

$$a^\dagger |\nu\rangle = \sum_{n=0}^{\infty} \sqrt{n+1}\, b_n |n+1\rangle = \sum_{k=1}^{\infty} \sqrt{k}\, b_{k-1} |k\rangle$$

$$= \nu |\nu\rangle = \sum_{k=0}^{\infty} \nu b_k |k\rangle$$

$$\Rightarrow \quad \nu b_k = \sqrt{k}\, b_{k-1}$$

$$\nu b_0 = 0$$

$$\nu b_1 = b_0 = 0 \quad , \ldots$$

All coefficients are zero.

$\therefore \; |\nu\rangle = 0$, not normalizable.

There is no state in Hilbert space (ket) that is an eigenstate of $a^\dagger$.

3) a) $\hat{\rho} = \hat{\rho}^{+}, \quad Tr\{\hat{\rho}\} = 1$

$Tr\{\hat{\rho}^2\} \leq 1 \quad (= \text{for pure states})$

$\Rightarrow c = b^*, \quad a + d = 1,$

$$Tr\{\hat{\rho}^2\} = Tr\left\{\begin{pmatrix} a^2 + |b|^2 & ab + bd \\ ab^* + db^* & |b|^2 + d^2 \end{pmatrix}\right\} = a^2 + d^2 + 2|b|^2$$

$$a^2 + d^2 + 2|b|^2 \leq 1 = (a+d)^2 = a^2 + d^2 + 2ad$$

$$|b|^2 \leq ad$$

b) $\langle \vec{S} \rangle = \frac{\hbar}{2} \langle \vec{\sigma} \rangle$

$$\langle \sigma_x \rangle = Tr\{\hat{\rho}\,\sigma_x\} = Tr\left\{\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} =$$

$$= Tr\left\{\begin{pmatrix} b & a \\ d & b^* \end{pmatrix}\right\} = b + b^*$$

$$\langle \sigma_y \rangle = Tr\{\hat{\rho}\,\sigma_y\} = Tr\left\{\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\}$$

$$= Tr\left\{\begin{pmatrix} ib & -ia \\ id & -ib^* \end{pmatrix}\right\} = i(b - b^*)$$

$$\langle \sigma_z \rangle = Tr\{\hat{\rho}\,\sigma_z\} = a - d$$

c) $\hat{\sigma}_i^2 = \mathbb{1}$

$$(\Delta S_i)^2 = \langle S_i^2 \rangle - \langle S_i \rangle^2 = \left(\frac{\hbar}{2}\right)^2 \left(1 - \langle \sigma_i \rangle^2\right)$$

$$(\Delta S_x)^2 = \left(\frac{\hbar}{2}\right)^2 \left(1 - (b + b^*)^2\right)$$

$$(\Delta S_y)^2 = \left(\frac{\hbar}{2}\right)^2 \left(1 + (b - b^*)^2\right)$$

d) $$\Delta S_x \Delta S_y \geq \frac{1}{2} \left| \langle [\hat{S}_x, \hat{S}_y] \rangle \right|$$

$$= \frac{\hbar}{2} \left| \langle S_z \rangle \right|$$

$$\Delta \sigma_x \Delta \sigma_y \geq \left| \langle \sigma_z \rangle \right|$$

$$\left(1 - \langle \sigma_x \rangle^2\right)\left(1 - \langle \sigma_y \rangle^2\right) \geq \left| \langle \sigma_z \rangle \right|^2 \quad ?$$

$$1 - \langle \sigma_x \rangle^2 - \langle \sigma_y \rangle^2 + \langle \sigma_x \rangle^2 \langle \sigma_y \rangle^2 \geq \langle \sigma_z \rangle^2 \quad ?$$

$$1 - \left(\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2\right) + \langle \sigma_x \rangle^2 \langle \sigma_y \rangle^2 \geq 0 \quad ?$$

$$1 - \langle \vec{\sigma} \rangle^2 + \langle \sigma_x \rangle^2 \langle \sigma_y \rangle^2 \geq 0 \quad ?$$

$$\langle \vec{\sigma} \rangle^2 = (b + b^*)^2 + i^2 (b - b^*)^2 + (a - d)^2$$

$$= 4|b|^2 + a^2 + d^2 - 2ad$$

$$1 - \langle \vec{\sigma} \rangle^2 = 4ad - 4|b|^2 \geq 0$$

$\therefore$ Generalized uncertainty relation holds.

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4) a) $|\psi\rangle$ is an eigestate of $\hat{J}_z = \hat{L}_z + \hat{S}_z$ with eigenvalue $\hbar - \frac{\hbar}{2} = \frac{\hbar}{2}$

$\mathcal{P}(J_z = \frac{\hbar}{2}) = 1$

b) $|\psi\rangle$ is a linear combination of states with $J = \frac{1}{2}$ and $J = \frac{3}{2}$

$$\hat{J}_+ |\psi\rangle = (\hat{L}_+ |l=1, m_l=1\rangle) |\downarrow\rangle + |l=1, m_l=1\rangle \hat{S}_+ |\downarrow\rangle$$

$$= 0 + \hbar |l=1, m_l=1\rangle |\uparrow\rangle$$

$$= \hbar |J = \tfrac{3}{2}, J_z = \tfrac{3}{2}\rangle$$

But $\hat{J}_+ |J\, m_j\rangle = \hbar \sqrt{J(J+1) - m_j(m_j+1)}\, |J, m_j+1\rangle$

so $\hat{J}_+ |\tfrac{3}{2}\, \tfrac{1}{2}\rangle = \hbar \sqrt{\tfrac{3}{2}(\tfrac{5}{2}) - \tfrac{1}{2}(\tfrac{3}{2})}\, |\tfrac{3}{2}\, \tfrac{3}{2}\rangle$

$$= \hbar \sqrt{3}\, |\tfrac{3}{2}\, \tfrac{3}{2}\rangle$$

Thus $|\psi\rangle = \frac{1}{\sqrt{3}} |\tfrac{3}{2}\, \tfrac{3}{2}\rangle + \beta |\tfrac{3}{2}\, \tfrac{1}{2}\rangle$

with $|\beta|^2 = \frac{2}{3}$.

$$\boxed{\begin{gathered} \mathcal{P}\left(J = \tfrac{3}{2}\right) = \frac{1}{3}, \quad \mathcal{P}\left(J = \tfrac{1}{2}\right) = \frac{2}{3} \\ \vec{J}^2 = \hbar^2 J(J+1). \end{gathered}}$$