Exercises for Physics 570A

Problem Set 10; Due in class Tuesday, December 10

1) Sankar 15.2.5

2) Sankar 15.2.6

3) The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction is

$$\hat{H} = A\vec{S}_1 \cdot \vec{S}_2 + \frac{eB}{mc}\left(\hat{S}_{1z} - \hat{S}_{2z}\right),$$

where $\vec{S}_1$ is the spin operator for the electron and $\vec{S}_2$ is the spin operator for the positron. Suppose the spin state of the system is given by $|\uparrow\rangle_1|\downarrow\rangle_2$.

a) Is this an eigenfunction of $\hat{H}$ in the limit $A \to 0$, $eB/mc \neq 0$? If it is, what is the energy eigenvalue. If not, what is $\langle\hat{H}\rangle$?

b) Solve the same problem when $eB/mc \to 0$, $A \neq 0$.

4) a) Consider a pure ensemble of identically prepared spin-1/2 systems. Suppose the expectation values $\langle S_x\rangle$ and $\langle S_z\rangle$ and the sign of $\langle S_y\rangle$ are known. Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle S_y\rangle$?

b) Consider a mixed ensemble of spin-1/2 systems. Suppose the ensemble averages $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ are all known. Construct the 2x2 density matrix that characterizes the ensemble.

5) a) Consider a system with a time-dependent Hamiltonian $\hat{H}(t)$. Prove that the time evolution of the density matrix $\hat{\rho}$ (in the Schrödinger picture) is given by

$$\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0).$$

b) Suppose that we have a pure ensemble at time $t = 0$. Prove that it cannot evolve into a mixed ensemble as long as the time evolution is governed by the Schrödinger equation.

6) Consider an ensemble of spin-1 systems. The density matrix is now a 3x3 matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ to characterize the ensemble completely?

7) Consider a system made up of two spin-1/2 particles. Observer A measures the spin components of particle 1 (S_{1x}, S_{1y}, and so on), while observer B measures the spin components of the other particle. Suppose the system is known to be in a spin-singlet state, that is $S_{\text{total}} = 0$.

a) What is the probability for observer A to obtain $S_{1z} = \hbar/2$ when observer B makes no measurement? Solve the same problem for $S_{1x} = \hbar/2$.

b) Observer B measures the z-component of the spin of particle 2, and finds the value $S_{2z} = \hbar/2$. What can we conclude about the outcome of observer A's measurement (i) if A measures S_{1z}; (ii) if A measures S_{1x}? Justify your answer.