Phys 570A HW 10 Solutions

1) **15.2.5** $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2 = \frac{3}{2}\hbar^2 + 2\vec{S}_1 \cdot \vec{S}_2$$

$$\vec{S}^2 = 2\hbar^2 \hat{P}_1 \quad \text{in this subspace}$$

$$\hat{P}_1 = \frac{\vec{S}^2}{2\hbar^2} = \frac{3}{4}\mathbb{1} + \frac{\vec{S}_1 \cdot \vec{S}_2}{\hbar^2}$$

$$\hat{P}_0 = \mathbb{1} - \hat{P}_1 = \frac{1}{4}\mathbb{1} - \frac{\vec{S}_1 \cdot \vec{S}_2}{\hbar^2} \quad \text{(2 d Hilbert space)}$$

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2) **15.2.6** $\vec{J}^2 = (\vec{L} + \vec{S})^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L} \cdot \vec{S}$

$$\vec{J}^2 = \hbar^2(\ell + \tfrac{1}{2})(\ell + \tfrac{3}{2})\hat{P}_+ + \hbar^2(\ell - \tfrac{1}{2})(\ell + \tfrac{1}{2})\hat{P}_-$$

$$= \hbar^2 \ell(\ell+1) + \frac{3}{4}\hbar^2 + 2\vec{L} \cdot \vec{S}$$

Also $\hat{P}_+ + \hat{P}_- = \mathbb{1}$ (in this subspace)

$$(\ell + \tfrac{1}{2})(\ell + \tfrac{3}{2})\hat{P}_+ + (\ell - \tfrac{1}{2})(\ell + \tfrac{1}{2})\hat{P}_- = \ell(\ell+1) + \frac{3}{4} + 2\vec{L} \cdot \vec{S}$$

After some algebra,

$$\hat{P}_+ = \frac{\ell + 1 + 2\vec{L} \cdot \vec{S}}{2\ell + 1}, \quad \hat{P}_- = \frac{\ell - 2\vec{L} \cdot \vec{S}}{2\ell + 1}$$

3) a) Yes, eigenvalue is $\frac{eB\hbar}{mc}$

b) No, $\langle \hat{H} \rangle = A \langle \vec{S}_1 \cdot \vec{S}_2 \rangle$

$$\vec{S}_1 \cdot \vec{S}_2 = \hat{S}_{1z}\hat{S}_{2z} + \frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)$$

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle = -\frac{\hbar^2}{4} \qquad \langle \hat{H} \rangle = -A\frac{\hbar^2}{4}$$

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4) a) $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\qquad |\alpha|^2 + |\beta|^2 = 1$

$$\left.\begin{array}{l} |\alpha|^2 + |\beta|^2 = 1 \\ \langle S_z \rangle = \frac{\hbar}{2}\left(|\alpha|^2 - |\beta|^2\right) \end{array}\right\} \begin{array}{l} |\alpha|^2 \\ |\beta|^2 \end{array}$$

$$\langle S_x \rangle = \frac{\hbar}{2}(\alpha^* \ \beta^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2}\left(\alpha^*\beta + \beta^*\alpha\right)$$

Let $\alpha = |\alpha| e^{ia}$, $\beta = |\beta| e^{ib}$, $a, b \in \mathbb{R}$.

$$\langle S_x \rangle = \frac{\hbar}{2}|\alpha||\beta|\left(e^{i(a-b)} + e^{i(b-a)}\right)$$

$$= \hbar |\alpha||\beta| \cos(a-b)$$

$$\langle S_y \rangle = \frac{\hbar}{2}(\alpha^* \ \beta^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = i\frac{\hbar}{2}\left(\alpha\beta^* - \beta\alpha^*\right)$$

$$= \hbar |\alpha||\beta| \sin(b-a)$$

$\sin(b-a) = \pm\sqrt{1-\cos^2(a-b)}$ Just need to know which root.

4) b) $\hat{\rho}^{\dagger} = \hat{\rho} = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$

$$\langle S_x \rangle = Tr\{\rho \sigma_x\} \frac{\hbar}{2} = \frac{\hbar}{2} Tr\left\{ \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$= \frac{\hbar}{2} tr\left\{ \begin{pmatrix} c & a \\ b & c^* \end{pmatrix} \right\} = \frac{\hbar}{2}(c + c^*)$$

$$\langle S_y \rangle = Tr\{\rho \sigma_y\} \frac{\hbar}{2} = \frac{\hbar}{2} Tr\left\{ \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}$$

$$= Tr\left\{ \begin{pmatrix} ic & -ia \\ ib & -ic^* \end{pmatrix} \right\} \frac{\hbar}{2} = \frac{i\hbar}{2}(c - c^*)$$

$$\langle S_z \rangle = \frac{\hbar}{2} Tr\{\rho \sigma_z\} = \frac{\hbar}{2} tr\left\{ \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$= \frac{\hbar}{2} Tr\left\{ \begin{pmatrix} a & -c \\ c^* & -b \end{pmatrix} \right\} = \frac{\hbar}{2}(a - b)$$

$$Tr\{\rho\} = a + b = 1$$

$$\langle \sigma_x \rangle = c + c^*$$

$$-i\langle \sigma_y \rangle = c - c^*$$

$$\langle \sigma_z \rangle = a - b$$

$$2a = 1 + \langle \sigma_z \rangle$$

$$2b = 1 - \langle \sigma_z \rangle$$

$$2c = \langle \sigma_x \rangle - i\langle \sigma_y \rangle$$

$$2c^* = \langle \sigma_x \rangle + i\langle \sigma_y \rangle$$

$$\hat{\rho} = \begin{pmatrix} \frac{1 + \langle \sigma_z \rangle}{2} & \frac{\langle \sigma_x \rangle - i \langle \sigma_y \rangle}{2} \\ \frac{\langle \sigma_x \rangle + i \langle \sigma_y \rangle}{2} & \frac{1 - \langle \sigma_z \rangle}{2} \end{pmatrix}$$

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5) a) $\hat{\rho} = \frac{1}{N} \sum_{k=1}^{N} |\psi^k(t)\rangle\langle\psi^k(t)|$

$|\psi^k(t)\rangle = \hat{U}(t, t_0) |\psi^k(t_0)\rangle$

$\therefore \hat{\rho}(t) = \frac{1}{N} \sum_{k=1}^{N} \hat{U}(t, t_0) |\psi^k(t_0)\rangle\langle\psi^k(t_0)| \hat{U}^\dagger(t, t_0)$

$= \hat{U}(t, t_0)\, \hat{\rho}(t_0)\, \hat{U}^\dagger(t, t_0)$

b) $\hat{\rho}(t=0) = |\psi_0\rangle\langle\psi_0|$

$\hat{\rho}(t) = \hat{U}(t, 0) |\psi_0\rangle\langle\psi_0| \hat{U}^\dagger(t, 0)$

$= |\psi(t)\rangle\langle\psi(t)|$

$(\hat{\rho}(t))^2 = \hat{\rho}(t) \quad \rightarrow$ still pure ensemble

6) $\hat{\rho} = \hat{\rho}^{\dagger}$ $\quad Tr\{\hat{\rho}\} = 1$

$\rho_{nm} = \rho^{*}_{mn}$ $\quad$ 3 real diagonal elements

-1 for constraint $Tr\{\rho\} = 1$

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3 complex elements above the diagonal

$= 6$ real parameters

$6 + 2 = 8$ real parameters

In addition to $\langle \vec{S} \rangle$, we would need 5 more expectation values to determine $\hat{\rho}$, For example

$\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, $\langle S_x S_y \rangle$, $\langle S_x S_z \rangle$, $\langle S_y S_z \rangle$.

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7) $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ $\quad S = 0$

a) $P(S_z = \frac{\hbar}{2}) = \frac{1}{2} = P(S_x = \frac{\hbar}{2})$

b) i) $S_{1z} = -\frac{\hbar}{2}$ $\quad$ ii) $P(S_{1x} = \frac{\hbar}{2}) = \frac{1}{2}$

$P(S_{1x} = -\frac{\hbar}{2}) = \frac{1}{2}$

Because $|\downarrow\rangle_1 = \alpha|+x\rangle_1 + \beta|-x\rangle_1$, with $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$.