Phys 570A HW 1 solutions

1.1.3) i) $f(0) = f(L) = 0$ yes

ii) $f(0) = f(L)$ yes

iii) $f(0) = 4$ no, not closed under addition, scaler mult.

1.1.4) $|1\rangle - 2|2\rangle = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix} = |3\rangle$

not linearly independent.

1.3.4) To prove: $|V+W| \leq |V| + |W|$

$$|V+W|^2 = |V|^2 + |W|^2 + \langle V|W\rangle + \langle W|V\rangle$$

$$= |V|^2 + |W|^2 + 2\mathrm{Re}\langle V|W\rangle$$

$$\leq |V|^2 + |W|^2 + 2|\langle V|W\rangle|$$

$$\leq |V|^2 + |W|^2 + 2|V||W| \quad \text{schwarz ineq.}$$

$$= (|V| + |W|)^2$$

Q.E.D.

1.6.2) $\Omega = \Omega^\dagger, \quad \Lambda = \Lambda^\dagger$

(1) $(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$ not clear

(2) $(\Omega\Lambda + \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger$

$= \Lambda\Omega + \Omega\Lambda = \Omega\Lambda + \Lambda\Omega$

$\rightarrow$ Hermitian

(3) $([\Omega, \Lambda])^\dagger = (\Omega\Lambda - \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger - \Omega^\dagger\Lambda^\dagger$

$= \Lambda\Omega - \Omega\Lambda = -[\Omega, \Lambda]$

$\rightarrow$ anti-Hermitian

(4) $(i[\Omega, \Lambda])^\dagger = -i(\Lambda\Omega - \Omega\Lambda) = i[\Omega, \Lambda]$

$\rightarrow$ Hermitian

1.6.3) Let U and V be unitary

$(UV)^\dagger = V^\dagger U^\dagger$

$(UV)^\dagger UV = V^\dagger U^\dagger U V = V^\dagger V = \mathbb{1}$ ✓

1.8.5) $\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

(1) $\Omega^\dagger\Omega = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(2) $\det\Omega = \cos^2\theta + \sin^2\theta$ $\quad \mathrm{Tr}\,\Omega = 2\cos\theta = \lambda_1 + \lambda_2$

$= \lambda_1\lambda_2 = 1$

$\lambda_1 = e^{i\alpha}$, $\lambda_2 = e^{-i\alpha}$ $\quad e^{i\alpha} + e^{-i\alpha} = 2\cos\theta$

$\Rightarrow \alpha = \pm\theta \quad$ Q.E.D.

(3) $\Omega|\pm\rangle = e^{\pm i\theta}|\pm\rangle$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = e^{\pm i\theta}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$(\cos\theta - e^{\pm i\theta})a + (\sin\theta)b = 0$$

$$(-\sin\theta)a + (\cos\theta - e^{\pm i\theta})b = 0$$

$$b = \frac{\cos\theta - (\cos\theta \pm i\sin\theta)}{\sin\theta} a = \mp i a$$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\langle -|+\rangle = \frac{1}{2}(1 \;\; -i)\begin{pmatrix} 1 \\ -i \end{pmatrix} = 0$$ ✓

(4) $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \qquad U^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

$$U^{+}\Omega U = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & e^{i\theta} \\ -ie^{-i\theta} & ie^{i\theta} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2e^{-i\theta} & 0 \\ 0 & 2e^{i\theta} \end{pmatrix}$$ ✓

1.8.10) $[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = 0$

$$\Omega\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$$\Lambda\Omega = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$\therefore$ Λ & Ω may be simultaneasly diagonalized.

$$0 = |\Omega - \omega \mathbb{1}| = \begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = \omega^2(\omega - 2)$$

$\omega = 0, 0, 2$

$$0 = |\Lambda - \lambda \mathbb{1}| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6$$

$= (3-\lambda)(2-\lambda)(-1-\lambda)$ $\qquad \lambda = 3, 2, -1$

Eigenvectors of Λ:

$\lambda = 3$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -a + b + c &= 0 \\ a - 3b - c &= 0 \\ \hline -2b &= 0 \end{aligned}$$

$a = b + c$

$b = 0$

$a = c$

$$|\lambda = 3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$a = b$

$$\begin{aligned} b + c &= 0 \\ a - 2b - c &= 0 \\ a - b &= 0 \end{aligned}$$

$c = a - 2b = -a$

$$|\lambda = 2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3a + b + c = 0 \qquad b = -2a$$

$$a + b - c = 0 \qquad c = a + b = -a$$

$$4a + 2b = 0 \qquad |\lambda = -1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Verify these are also eigenvectors of Ω:

$$\Omega |\lambda = 3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2 |\lambda = 3\rangle$$

$$\Rightarrow \omega = 2$$

$$\Omega |\lambda = 2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 |\lambda = 2\rangle$$

$$\Rightarrow \omega = 0$$

$$\Omega |\lambda = -1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 |\lambda = -1\rangle$$

$$\Rightarrow \omega = 0$$

1.9.1) $f(\Omega)|\omega\rangle = \omega|\omega\rangle$, $\Omega|\omega\rangle = \omega|\omega\rangle$

$$f(\Omega)|\omega\rangle = \sum_{n=0}^{\infty} \Omega^n |\omega\rangle = \sum_{n=0}^{\infty} \omega^n |\omega\rangle$$

$$= \frac{1}{1-\omega}|\omega\rangle \quad \text{if } |\omega| < 1$$

Then $f(\Omega) = (1-\Omega)^{-1}$ provided all eigenvalues of Ω satisfy $|\omega| < 1$.