Phys 570A HW 2 solutions

1.10.1) $\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$

$\int_{-\infty}^{\infty} dx\, \delta(ax) f(x) = ?$ Let $y = ax$

$$= \int_{-\infty/a}^{\infty/a} \frac{dy}{a}\, \delta(y) f(y/a) = \frac{1}{|a|}\int_{-\infty}^{\infty} dy\, \delta(y) f(y/a) = \frac{f(0)}{|a|}$$

$\therefore \quad \delta(ax) = \dfrac{\delta(x)}{|a|}$

1.10.2) $\delta(f(x)) = ?$ Let x_i, $i = 1, \ldots, N$

be zeros of $f(x)$: $f(x_i) = 0$.

Near x_i, $f(x) \simeq f'(x_i)(x - x_i)$.

Using result from (1.10.1),

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

1.10.3) $\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx}\theta(x - x')$

$$= f(x)\theta(x-x')\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f'(x)\theta(x - x')$$

$$= f(\infty) - \int_{x'}^{\infty} dx\, f'(x) = f(\infty) - f(\infty) + f(x') = f(x')$$

$$\therefore \quad \frac{d}{dx}\theta(x-x') = \delta(x-x').$$

4.2.1) (1) $L_z = 1, 0, -1$

(2) $|L_z=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\langle L_x\rangle = (1\ 0\ 0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1\,0\,0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle L_x^2\rangle = ? \qquad L_x^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\langle L_x^2\rangle = \frac{1}{2}$$

$$(\Delta L_x)^2 = \langle L_x^2\rangle - \langle L_x\rangle^2 = \frac{1}{2}$$

(3) See solution, p. 665

(4) $L_z = -1$ $\qquad P(L_x=1) = |\langle L_x=1|L_z=-1\rangle|^2 = \frac{1}{4}$

$P(L_x=0) = \frac{1}{2}$ $\qquad P(L_x=-1) = \frac{1}{4}$

(5) $L_z^2 = +1$ Probability $= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

After measurement $|\psi\rangle \to \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

L_z measured: $P(L_z=1) = \frac{2}{3}$, $P(L_z=-1) = \frac{1}{3}$

(6)

$$|\langle L_z=1|\psi\rangle|^2 = \frac{1}{4}$$

$$|\langle L_z=0|\psi\rangle|^2 = \frac{1}{2}$$

$$|\langle L_z=-1|\psi\rangle|^2 = \frac{1}{4}$$

$$\therefore |\psi\rangle = \sqrt{\tfrac{1}{4}}\, e^{i\delta_1} |L_z=1\rangle + \sqrt{\tfrac{1}{2}}\, e^{i\delta_2} |L_z=0\rangle + \sqrt{\tfrac{1}{4}}\, e^{i\delta_3} |L_z=-1\rangle$$

Factors $e^{i\delta_i}$ matter due to interferences which depend on e.g. $\delta_1 - \delta_2$.

$$P(L_x=0) = |\langle L_x=0|\psi\rangle|^2$$

$$\langle L_x=0|\psi\rangle = -\frac{1}{\sqrt{2}}\frac{1}{2}e^{i\delta_1} + \frac{1}{\sqrt{2}}\frac{1}{2}e^{i\delta_3}$$

$$P(L_x=0) = \frac{1}{2}\sin^2\left(\frac{\delta_1-\delta_3}{2}\right)$$

4.2.2) $\langle P_x \rangle = \int dx\, \psi(x) \frac{\hbar}{i} \psi'(x)$

$$\langle P_x \rangle = \frac{\hbar}{2i} \int_a^b dx\, \frac{d}{dx}(\psi(x))^2 = \frac{\hbar}{2i}\, \psi^2(x)\Big|_a^b$$

If $(a,b) = (-\infty, \infty)$, then $\langle P_x \rangle = 0$ because $\psi(\pm\infty) = 0$. If ψ obeys Dirichlet b.c.s on (a,b), then $\langle P_x \rangle = 0$. If $\psi(x)$ obeys PBCs on (a,b), then $\langle P_x \rangle = 0$.

5.3.1) $\hat{H} = \hat{T} + \hat{V}$

$$\hat{H}^\dagger = \hat{T}^\dagger + \hat{V}^\dagger = \hat{T} + V_r + iV_i \neq \hat{H}$$

$$\left[i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right]\psi^*$$

$$-\left[-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V^*\psi^*\right]\psi$$

$$i\hbar \frac{\partial}{\partial t}|\psi|^2 = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) - 2iV_i|\psi|^2$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J} - \frac{2V_i}{\hbar}\rho$$

$$\frac{d}{dt}P = \int d^3x \, \frac{\partial \rho}{\partial t} = -\int d^3x \, \nabla \cdot \vec{J} - \frac{2V_i}{\hbar} P$$

$$\frac{dP}{dt} = -\cancelto{0}{\int \vec{J} \cdot d\vec{S}} - \frac{2V_i}{\hbar} P(t)$$

solution $\quad P(t) = P(0)\, e^{-\frac{2V_i}{\hbar} t}$

5.3.4) $\psi = A e^{i\frac{px}{\hbar}} + B e^{-i\frac{px}{\hbar}}$

$$j_x = \frac{1}{m} \mathrm{Re}\left\{\psi^* \frac{\hbar}{i} \frac{d}{dx} \psi\right\}$$

$$= \frac{p}{m} \mathrm{Re}\left\{\left(A^* e^{-i\frac{px}{\hbar}} + B^* e^{i\frac{px}{\hbar}}\right)\left(A e^{i\frac{px}{\hbar}} - B e^{-i\frac{px}{\hbar}}\right)\right\}$$

$$= \frac{p}{m} \mathrm{Re}\left(|A|^2 - |B|^2 + B^*A - A^*B\right)$$

$$= \frac{p}{m}\left(|A|^2 - |B|^2\right)$$