1) $\langle E \rangle = \frac{1}{2m} \langle p_x^2 \rangle + \frac{m\omega^2}{2} \langle x^2 \rangle$

$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \leq \langle x^2 \rangle$

$(\Delta p_x)^2 \leq \langle p_x^2 \rangle$

$\Rightarrow \quad \langle E \rangle \geq \frac{1}{2m} (\Delta p_x)^2 + \frac{m\omega^2}{2} (\Delta x)^2$

Also $\quad (\Delta x)^2 (\Delta p_x)^2 \geq \left(\frac{\hbar}{2}\right)^2$

$\langle E \rangle \geq \frac{1}{2m} \frac{\hbar^2}{4(\Delta x)^2} + \frac{m\omega^2}{2} (\Delta x)^2$

minimize over Δx:

$0 = \frac{\partial \langle E \rangle}{\partial \Delta x} = -\frac{1}{2m} \frac{\hbar^2}{2(\Delta x)^3} + m\omega^2 \Delta x$

$(\Delta x)^4 = \frac{\hbar^2}{4m^2\omega^2} \qquad (\Delta x)^2 = \frac{\hbar}{2m\omega}$

$\langle E \rangle \geq \frac{\hbar^{\cancel{2}}}{8\cancel{m}} \frac{2\cancel{m}\omega}{\cancel{\hbar}} + \frac{m\omega^{\cancel{2}}}{2} \frac{\hbar}{2\cancel{m}\cancel{\omega}} = \frac{\hbar\omega}{2}$

This is the ground-state energy.

2) Let $|\psi\rangle = \alpha|1\rangle + \beta|2\rangle$

$\hat{H}|\psi\rangle = E|\psi\rangle$ $\qquad \langle 1|\hat{H}|\psi\rangle = E\langle 1|\psi\rangle$

$\qquad\qquad\qquad\qquad \langle 2|\hat{H}|\psi\rangle = E\langle 2|\psi\rangle$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$0 = \det\begin{pmatrix} H_{11}-E & H_{12} \\ H_{12} & H_{22}-E \end{pmatrix} = (E-H_{11})(E-H_{22}) - H_{12}^2$$

$$0 = E^2 - (H_{11}+H_{22})E - H_{12}^2 + H_{11}H_{22}$$

$$E_{\pm} = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{11}+H_{22}}{2}\right)^2 + H_{12}^2 - H_{11}H_{22}}$$

$$\boxed{E_{\pm} = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{11}-H_{22}}{2}\right)^2 + H_{12}^2}}$$

Let $\bar{\varepsilon} = \dfrac{H_{11} + H_{22}}{2}$, $\dfrac{H_{22} - H_{11}}{2} = \Delta$,

$\dfrac{\Delta}{H_{12}} = x$, $H_{12} = V$.

$$E_{\pm} = \bar{\varepsilon} \pm |V| \sqrt{1 + x^2}$$

$$(H_{11} - E_{\pm})\alpha_{\pm} + H_{12}\beta_{\pm} = 0$$

$$\beta_{\pm} = \frac{E_{\pm} - H_{11}}{H_{12}} \alpha_{\pm}$$

$$\beta_{\pm} = \frac{\Delta \pm |V|\sqrt{1+x^2}}{V} \alpha_{\pm} = \left(x \pm \sqrt{1+x^2}\right)\alpha_{\pm}$$

Eigenfunctions:

$$|\psi_{\pm}\rangle = \alpha_{\pm}\left(|1\rangle + \left(x \pm \sqrt{1+x^2}\right)|2\rangle\right)$$

Normalization:

$$1 = \langle \psi_\pm | \psi_\pm \rangle = |\alpha_\pm|^2 \left(1 + \left(x \pm \sqrt{1+x^2}\right)^2\right)$$

$$\boxed{\alpha_\pm = \frac{1}{\sqrt{1 + \left(x \pm \sqrt{1+x^2}\right)^2}}}$$

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3) Since $\{|a', b'\rangle\}$ forms a complete basis in Hilbert space, a general state can be written

$$|\psi\rangle = \sum_{a', b'} C_{a'b'} |a' b'\rangle$$

$$[\hat{A}, \hat{B}] |\psi\rangle = \sum_{a', b'} C_{a'b'} [\hat{A}, \hat{B}] |a' b'\rangle$$

But $[\hat{A}, \hat{B}] |a' b'\rangle = (a'b' - b'a') |a' b'\rangle = 0$

So $[\hat{A}, \hat{B}] |\psi\rangle = 0 \quad \forall |\psi\rangle \in \mathcal{H}.$

$$\Rightarrow [\hat{A}, \hat{B}] = 0$$

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4) Suppose $A|\psi\rangle = a|\psi\rangle$ and $B|\psi\rangle = b|\psi\rangle$

$$\{A, B\} |\psi\rangle = (AB + BA) |\psi\rangle = 2ab |\psi\rangle = 0$$

This is possible if $a = 0$ or $b = 0$.

5) $[A_1, A_2] \neq 0$, $[A_1, H] = 0$, $[A_2, H] = 0$.

Suppose $H|\nu\rangle = E_\nu|\nu\rangle$ with E_ν distinct eigenvalues (no degeneracy). Then, from the theorem proved at the beginning of lecture 12, $|\nu\rangle$ must also be an eigenstate of A_1 and A_2.

$$A_1|\nu\rangle = a_\nu|\nu\rangle \quad , \quad A_2|\nu\rangle = b_\nu|\nu\rangle .$$

$$[A_1, A_2]|\nu\rangle = (a_\nu b_\nu - b_\nu a_\nu)|\nu\rangle = 0 .$$ But a general state can be written

$$|\psi\rangle = \sum_\nu C_\nu |\nu\rangle .$$

$$[A_1, A_2]|\psi\rangle = \sum_\nu C_\nu [A_1, A_2]|\nu\rangle = 0 .$$

This is a contradiction, since $[A_1, A_2] \neq 0$.

$\therefore$ The energy eigenstates must in general be degenerate.

An exception can occur if $a_\nu = 0$ and/or $b_\nu = 0$. For instance, the $l=0$ states in a central potential have $L_x = L_y = L_z = 0$ even though $[L_x, L_y] \neq 0$, etc.