Phys 570A HW 5
Solutions

1) a) $\{A, B\} = \dfrac{\partial A}{\partial x}\dfrac{\partial B}{\partial p_x} - \dfrac{\partial A}{\partial p_x}\dfrac{\partial B}{\partial x}$

$\{x, F(p_x)\} = F'(p_x)$

b) $\left[x, e^{i\frac{p_x a}{\hbar}}\right] = i\hbar \left\{x, e^{i\frac{p_x a}{\hbar}}\right\} = i\hbar \dfrac{ia}{\hbar} e^{i\frac{p_x a}{\hbar}}$

$$= -a\, e^{i\frac{p_x a}{\hbar}}$$

c) $\hat{x}\, e^{i\frac{\hat{p}_x a}{\hbar}} |x'\rangle = \left[\hat{x}, e^{i\frac{\hat{p}_x a}{\hbar}}\right]|x'\rangle + e^{i\frac{\hat{p}_x a}{\hbar}}\hat{x}|x'\rangle$

$$= (x' - a)\, e^{i\frac{\hat{p}_x a}{\hbar}} |x'\rangle$$

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2) Let $|\phi\rangle = e^{i\frac{\vec{p}\cdot\vec{a}}{\hbar}} |\psi\rangle$

$$\langle\phi|\vec{x}|\phi\rangle = \langle\psi| e^{-i\frac{\vec{p}\cdot\vec{a}}{\hbar}}\, \vec{x}\, e^{i\frac{\vec{p}\cdot\vec{a}}{\hbar}} |\psi\rangle$$

$$= \langle\psi| e^{-i\frac{\vec{p}\cdot\vec{a}}{\hbar}} e^{i\frac{\vec{p}\cdot\vec{a}}{\hbar}}\, \vec{x}\, |\psi\rangle$$
$$- \vec{a}\, \langle\psi| e^{-i\frac{\vec{p}\cdot\vec{a}}{\hbar}} e^{i\frac{\vec{p}\cdot\vec{a}}{\hbar}} |\psi\rangle$$

$$= \langle\psi|\vec{x}|\psi\rangle - \vec{a}$$

$$\langle\phi|\vec{p}|\phi\rangle = \langle\psi| e^{-i\frac{\vec{p}\cdot\vec{a}}{\hbar}}\, \vec{p}\, e^{i\frac{\vec{p}\cdot\vec{a}}{\hbar}} |\psi\rangle = \langle\psi|\vec{p}|\psi\rangle$$

3) $\exp\left(i\frac{\hat{x}Q}{\hbar}\right)$ is a momentum translation operator.

$$\hat{p}_x \exp\left(i\frac{\hat{x}Q}{\hbar}\right)|p'\rangle = p' \exp\left(i\frac{\hat{x}Q}{\hbar}\right)|p'\rangle + \left[\hat{p}_x, \exp\left(i\frac{\hat{x}Q}{\hbar}\right)\right]|p'\rangle$$

But $\left[\hat{p}_x, \exp\left(i\frac{\hat{x}Q}{\hbar}\right)\right] = -i\hbar\frac{\partial}{\partial\hat{x}}\exp\left(i\frac{\hat{x}Q}{\hbar}\right)$

$$= +Q\exp\left(i\frac{\hat{x}Q}{\hbar}\right)$$

So $\hat{p}_x \exp\left(i\frac{\hat{x}Q}{\hbar}\right)|p'\rangle = (p'+Q)\exp\left(i\frac{\hat{x}Q}{\hbar}\right)|p'\rangle$

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4) For a free particle, one has

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}_x(0)\,t}{m},$$

so $[\hat{x}(t), \hat{x}(0)] = \frac{t}{m}[\hat{p}_x(0), \hat{x}(0)]$

$$= -\frac{i\hbar t}{m}$$

$$\Delta x(t)\,\Delta x(0) \geq \frac{1}{2}\left|\left\langle -\frac{i\hbar t}{m}\right\rangle\right| = \frac{\hbar t}{2m}$$

$$\Delta x(t) \geq \frac{\hbar t}{2m\,\Delta x(0)}$$

5) $$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}_x(0)\,t}{m}$$

$$\hat{p}_x(t) = \hat{p}_x(0)$$

Let $$\Delta\hat{x}(t) \equiv \hat{x}(t) - \langle \hat{x}(t) \rangle$$

$$= \hat{x}(0) - \langle \hat{x}(0) \rangle + \left(\hat{p}_x(0) - \langle \hat{p}_x(0) \rangle\right)\frac{t}{m}$$

$$\Delta p_x(t) \equiv \hat{p}_x(t) - \langle \hat{p}_x(t) \rangle = \hat{p}_x(0) - \langle \hat{p}_x(0) \rangle$$

$$\langle (\Delta p_x(t))^2 \rangle = \langle (\hat{p}_x(0) - \langle p_x(0) \rangle)^2 \rangle = (\Delta p_x(0))^2$$

$$\langle (\Delta x(t))^2 \rangle = \left\langle \left(\hat{x}(0) - \langle x(0) \rangle + (\hat{p}_x(0) - \langle p_x(0) \rangle)\frac{t}{m} \right)^2 \right\rangle$$

$$= \langle (\Delta x(0))^2 \rangle + \langle (\Delta p_x(0))^2 \rangle \frac{t^2}{m^2}$$

$$+ \frac{t}{m} \underbrace{\left\langle \left[\hat{x}(0) - \langle x(0) \rangle, \hat{p}_x(0) - \langle p_x(0) \rangle \right]_+ \right\rangle}_{\to 0 \text{ using hint}}$$

$$= \langle (\Delta x(0))^2 \rangle + \frac{\hbar^2 t^2}{4m^2 \langle (\Delta x(0))^2 \rangle}$$