Phys 570A HW6 Solutions

$$|\psi(t=0)\rangle = \exp(-i\hat{P}_x d/\hbar)\,|0\rangle$$ (use "d" instead of "a")

a) $$e^{-i\hat{P}_x d/\hbar}|0\rangle = \exp\left(d\sqrt{\frac{m\omega}{2\hbar}}(a^+-a)\right)|0\rangle$$

using $\hat{P}_x = i\sqrt{\frac{m\hbar\omega}{2}}(a^+-a)$.

Use $e^{A+B} = e^A e^B e^{-[A,B]/2}$, $[a,a^+]=1$:

$$e^{-i\hat{P}_x d/\hbar}|0\rangle = e^{d\sqrt{\frac{m\omega}{2\hbar}}a^+} e^{-d\sqrt{\frac{m\omega}{2\hbar}}a} e^{-d^2\frac{m\omega}{4\hbar}}|0\rangle$$

$$= e^{-\frac{m\omega d^2}{4\hbar}} e^{\sqrt{\frac{m\omega d^2}{2\hbar}}a^+}|0\rangle$$

(because $e^{-\lambda a}|0\rangle = |0\rangle$).

This is of the general form

$$e^{-|\lambda|^2/2} e^{\lambda a^+}|0\rangle, \text{ with } \lambda = \sqrt{\frac{m\omega d^2}{2\hbar}}.$$

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b) $|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^+}|0\rangle$. To show

$a|\lambda\rangle = \lambda|\lambda\rangle$. $\quad a|\lambda\rangle = e^{-|\lambda|^2/2} a\, e^{\lambda a^+}|0\rangle$

$$e^{\lambda a^+}|0\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n (a^+)^n}{n!}|0\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

$$a|\lambda\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n \, a \, |n\rangle}{\sqrt{n!}}$$

$$= e^{-|\lambda|^2/2} \sum_{n=1}^{\infty} \frac{\lambda^n \sqrt{n} \, |n-1\rangle}{\sqrt{n!}}$$

$$= \lambda e^{-|\lambda|^2/2} \sum_{n'=0}^{\infty} \frac{\lambda^{n'} \, |n'\rangle}{\sqrt{n'!}} = \lambda|\lambda\rangle \checkmark$$

$$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} = 1 \checkmark$$

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c) $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$

$$\hat{U}(t) = e^{-i\frac{\hat{H}t}{\hbar}} = e^{-i\omega t(a^\dagger a + \frac{1}{2})}$$

$$\hat{U}(t)|\Psi(0)\rangle = e^{-|\lambda|^2/2} e^{-i\omega t(a^\dagger a + \frac{1}{2})} \sum_{n=0}^{\infty} \frac{\lambda^n \, |n\rangle}{\sqrt{n!}}$$

$$= e^{-|\lambda|^2/2} e^{-i\frac{\omega t}{2}} \sum_{n=0}^{\infty} \frac{\lambda^n e^{-in\omega t} \, |n\rangle}{\sqrt{n!}}$$

$$= e^{-|\lambda|^2/2} e^{-i\frac{\omega t}{2}} e^{\lambda(t) a^\dagger} |0\rangle, \quad \lambda(t) = \lambda e^{-i\omega t}$$

$$= e^{-i\frac{\omega t}{2}} |\lambda(t)\rangle$$

d) $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a^{+} + a\right)$, $\hat{p}_x = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^{+} - a\right)$

$a|\lambda\rangle = \lambda|\lambda\rangle$ $\quad\quad$ $\langle\lambda|a^{+} = \lambda^{*}\langle\lambda|$

$$\langle x(t)\rangle = \langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle\lambda(t)|\,a^{+} + a\,|\lambda(t)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\lambda^{*}(t) + \lambda(t)\right)$$

$$= \sqrt{\frac{2\hbar\lambda^2}{m\omega}}\cos\omega t = d\cos\omega t$$

$$\langle p_x(t)\rangle = \langle\psi(t)|\,\hat{p}_x\,|\psi(t)\rangle$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\langle\lambda(t)|\,a^{+} - a\,|\lambda(t)\rangle$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\left(\lambda^{*}(t) - \lambda(t)\right)$$

$$= -\sqrt{2m\hbar\omega}\,\lambda\sin\omega t = -m\omega d\sin\omega t$$

$$= m\frac{d}{dt}\langle x(t)\rangle$$

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e) $(\Delta x(t))^2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2$

$(\Delta p_x(t))^2 = \langle p_x^2(t)\rangle - \langle p_x(t)\rangle^2$

$$\langle x^2(t)\rangle = \frac{\hbar}{2m\omega}\langle\lambda(t)|(a^\dagger + a)^2|\lambda(t)\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\lambda(t)|(a^\dagger)^2 + a^2 + a^\dagger a + a a^\dagger|\lambda(t)\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\lambda(t)|(a^\dagger)^2 + a^2 + 2a^\dagger a + 1|\lambda(t)\rangle$$

$$= \frac{\hbar}{2m\omega}\left[(\lambda^*(t))^2 + (\lambda(t))^2 + 2\lambda(t)\lambda^*(t) + 1\right]$$

$$\langle x(t)\rangle^2 = \frac{\hbar}{2m\omega}\left[\lambda^*(t) + \lambda(t)\right]^2$$

$$\boxed{(\Delta x(t))^2 = \frac{\hbar}{2m\omega} = (\Delta x(0))^2}$$

$$\langle p_x^2(t)\rangle = -\frac{m\hbar\omega}{2}\langle\lambda(t)|(a^\dagger - a)^2|\lambda(t)\rangle$$

$$= -\frac{m\hbar\omega}{2}\langle\lambda(t)|(a^\dagger)^2 + a^2 - a^\dagger a - a a^\dagger|\lambda(t)\rangle$$

$$= \frac{m\hbar\omega}{2}\left[2\lambda^*(t)\lambda(t) + 1 - (\lambda^*(t))^2 - (\lambda(t))^2\right]$$

$$(\Delta p_x(t))^2 = \frac{m\hbar\omega}{2}\left[1 - (\lambda^*(t) - \lambda(t))^2 + (\lambda^*(t) - \lambda(t))^2\right]$$

$$= \frac{m\hbar\omega}{2} = (\Delta p_x(0))^2$$

$$\Delta x(t)\,\Delta p_x(t) = \left(\frac{\hbar}{2m\omega}\frac{m\hbar\omega}{2}\right)^{1/2} = \frac{\hbar}{2}$$