# Physics 570A HW 7
## solutions

1) Choice of gauge: $A_x = \frac{\Phi}{L}$

$\vec{B} = \nabla \times \vec{A} = 0$ (can verify in cylindrical coordinates)

a)

Ansatz: $\boxed{\psi(x) = \frac{1}{\sqrt{L}} e^{ikx}}$

PBC: $\psi(x+L) = \psi(x) \Rightarrow kL = 2\pi n$

$$\boxed{k_n = \frac{2\pi n}{L}}$$

$$H e^{ikx} = \frac{1}{2m}\left(\hbar k - \frac{q\Phi}{cL}\right)^2 e^{ikx}$$

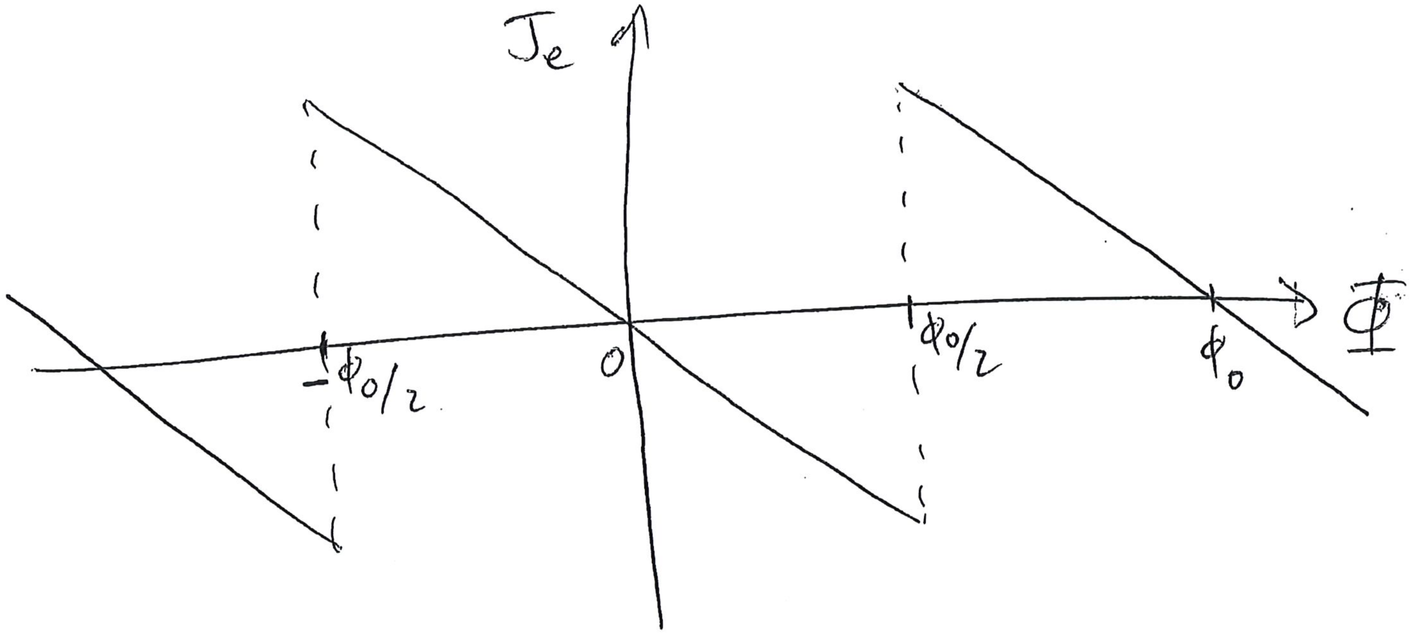
$$\boxed{E_n = \frac{h^2}{2mL^2}\left(n - \Phi/\phi_0\right)^2, \quad \text{where } \phi_0 = hc/q}$$

b) $J_e^{(n)} = \frac{e}{m} \mathrm{Re}\left\{ \psi^* \left( \frac{\hbar}{i}\frac{d}{dx} - \frac{e}{c}A_x \right) \psi \right\}$

$$= \frac{e}{mL}\left( \hbar k_n - \frac{e}{c}\frac{\Phi}{L} \right)$$

For $-\frac{\phi_0}{2} < \Phi < \frac{\phi_0}{2}$, the ground state has $n = 0$.

$$J_e^{(0)} = -\frac{e^2}{mcL^2}\Phi \ , \quad -\frac{\phi_0}{2} < \Phi < \frac{\phi_0}{2}$$



# Physics 570A Problem Set 7

## Solutions

2) $$[v_x, v_y] = \frac{1}{m^2}\left[p_x - \frac{q}{c}A_x,\ p_y - \frac{q}{c}A_y\right]$$

$$= \frac{1}{m^2}\left(-\frac{q}{c}\right)\left([p_x, A_y] + [A_x, p_y]\right)$$

Recall: $[p_i, f(x,y,z)] = -i\hbar \frac{\partial f}{\partial x_i}$

$$\Rightarrow [v_x, v_y] = \frac{iq\hbar}{m^2 c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \frac{iq\hbar}{m^2 c} B_z$$

Similarly,

$$[v_y, v_z] = \frac{iq\hbar}{m^2 c}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) = \frac{iq\hbar}{m^2 c} B_x$$

$$[v_z, v_x] = \frac{iq\hbar}{m^2 c}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = \frac{iq\hbar}{m^2 c} B_y$$

3) $H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2$

$$= \frac{m\vec{V}^2}{2} = \frac{m V_x^2}{2} + \frac{m V_y^2}{2} + \frac{m V_z^2}{2}$$

$V_z$ commutes with $V_x$ and $V_y$ if we take $\vec{B} = B\hat{z}$.

$$E = \langle H \rangle = \frac{m}{2}\langle V_x^2 + V_y^2\rangle + \frac{m}{2}\langle V_z^2\rangle$$

$$\geq \frac{m}{2}\langle V_x^2 + V_y^2\rangle$$

$$\Delta V_y \, \Delta V_x \geq \frac{1}{2}\left|\frac{e\hbar B}{m^2 c}\right| = \frac{\hbar\Omega}{2m},$$

where $\Omega = eB/mc$ = cyclotron frequency.

Now $\langle V_x^2\rangle = (\Delta V_x)^2 + \langle V_x\rangle^2 \geq (\Delta V_x)^2$

and $\langle V_y^2\rangle \geq (\Delta V_y)^2$, so

$$E \geq \frac{m}{2}\left[(\Delta v_x)^2 + (\Delta v_y)^2\right].$$

But $\Delta v_y \geq \dfrac{\hbar\Omega}{2m\,\Delta v_x}$, so

$$E \geq \frac{m}{2}(\Delta v_x)^2 + \frac{\hbar^2\Omega^2}{8m(\Delta v_x)^2} \quad (\star)$$

The expression on the r.h.s. of $(\star)$ takes a minimum when

$$0 = m\,\Delta v_x - \frac{\hbar^2\Omega^2}{4m(\Delta v_x)^3}$$

or $(\Delta v_x)^4 = \left(\dfrac{\hbar\Omega}{2m}\right)^2$

$$(\Delta v_x)^2 = \frac{\hbar\Omega}{2m}$$

$$\Rightarrow E \geq \frac{\hbar\Omega}{4} + \frac{\hbar\Omega}{4} = \frac{\hbar\Omega}{2}$$