Phys 570A HW 8

solutions

1) $\langle x', t' | x, t \rangle = \int \mathcal{D}[x(t)] \, e^{\frac{i}{\hbar}\int_t^{t'} L(t'')dt''}$

$$= \lim_{N\to\infty} \left(\frac{mN}{i\hbar(t'-t)}\right)^{N/2} \int \prod_{n=1}^{N-1} dx_n \, e^{i\frac{S}{\hbar}},$$

where $x_0 = x$, $x_N = x'$, and

$$t_n = t + n\frac{(t'-t)}{N}.$$

$$\int_{t_{n-1}}^{t_n} L(t)\,dt = \frac{m(x_n - x_{n-1})^2}{2\Delta t} + V(x_n)\Delta t + \mathcal{O}(\Delta t)^2$$

For a free particle $V(x) = 0$.

i) N = 1 $\quad \langle x', t' | x, t\rangle = \left(\frac{m}{i\hbar(t'-t)}\right)^{1/2} e^{i\frac{(x'-x)^2}{2\hbar(t'-t)}}$

ii) N = 2 $\quad \int L(t)\,dt = \frac{m(x'-x_1)^2}{2\Delta t} + \frac{m(x_1-x)^2}{2\Delta t}$

$$\langle x', t' | x, t\rangle = \frac{2m}{i\hbar(t'-t)} \int_{-\infty}^{\infty} dx_1 \, e^{\frac{im}{2\hbar\Delta t}\left[(x'-x_1)^2 + (x_1-x)^2\right]}$$

$$(x'-x_1)^2 + (x_1-x)^2 = x'^2 + x^2 + 2x_1^2 - 2x_1(x'+x)$$

$$= 2\left(x_1 - \frac{x'+x}{2}\right)^2 + \frac{(x'-x)^2}{2}$$

$$\langle x',t' | x,t\rangle = \frac{2m}{i\hbar(t'-t)} e^{\frac{im}{4\hbar\Delta t}(x'-x)^2} \underbrace{\int_{-\infty}^{\infty} d\xi\, e^{2\frac{im}{2\hbar\Delta t}\xi^2}}_{\sqrt{\frac{\pi\hbar i\Delta t}{m}}}$$

$$= \sqrt{\frac{m}{i\hbar 2\Delta t}}\, e^{\frac{im(x'-x)^2}{2\hbar 2\Delta t}}$$

$$= \sqrt{\frac{m}{i\hbar(t'-t)}}\, e^{\frac{im(x'-x)^2}{2\hbar(t'-t)}}$$ (same as for $N=1$)

(ii) In general, the integral over x_1 involves

$$\frac{im}{2\hbar\Delta t}\left[(x_2-x_1)^2 + (x_1-x)^2\right] = \frac{im}{2\hbar\Delta t}\left[2\left(x_1 - \frac{x_2+x}{2}\right)^2 + \frac{(x_2-x)^2}{2}\right]$$

$$\int_{-\infty}^{\infty} dx_1\, e^{\frac{im}{2\hbar\Delta t}\left[2\left(x_1 - \frac{x_2+x}{2}\right)^2 + \frac{(x_2-x)^2}{2}\right]} = e^{\frac{im(x_2-x)^2}{4\hbar\Delta t}} \sqrt{\frac{i h \Delta t}{2m}}$$

Then, the integral over x_2 involves

$$\frac{im}{2\hbar\Delta t}\left[(x_3 - x_2)^2 + \frac{(x_2 - x)^2}{2}\right]$$

$$= \frac{3im}{4\hbar\Delta t}\left(x_2 - \frac{2x_3 + x}{3}\right)^2 + \frac{im}{2\hbar 3\Delta t}(x_3 - x)^2$$

The x_2 integral gives

$$\sqrt{\frac{4\pi i\hbar\Delta t}{3m}}\; e^{\frac{im}{2\hbar}\frac{(x_3 - x)^2}{3\Delta t}}$$

Combining all the factors, for $N=3$ we would get

$$\left(\frac{m}{i\hbar 3\Delta t}\right)^{1/2} e^{\frac{im}{2\hbar}\frac{(x_3 - x)^2}{3\Delta t}},$$ where $x_3 = x'$ and $3\Delta t = t' - t$

(same as for $N=1$ and $N=2$).

The pattern becomes clear: The integrals up to x_{n-1} give

$$\left(\frac{i\hbar\Delta t}{m}\right)^{\frac{n-1}{2}} \frac{1}{\sqrt{n}}\; e^{\frac{im}{2\hbar}\frac{(x_n - x)^2}{n\Delta t}} \qquad (1).$$

Then the integral over x_n involves

$$\frac{im}{2\hbar\Delta t}\left[(x_{n+1}-x_n)^2 + \frac{(x_n-x)^2}{n}\right]$$

$$= \frac{im}{2\hbar n\Delta t}\left[(n+1)\left(x_n - \frac{n x_{n+1}+x}{n+1}\right)^2 + \frac{n}{n+1}(x_{n+1}-x)^2\right].$$

The x_n integral gives

$$\sqrt{\frac{ih n\Delta t}{(n+1)m}}\; e^{\frac{im\,(x_{n+1}-x)^2}{2\hbar(n+1)\Delta t}}$$

combining with the prefactor from the integrals up to x_{n-1} integral gives

$$\left(\frac{ih\Delta t}{m}\right)^{\frac{n}{2}}\frac{1}{\sqrt{n+1}}\; e^{\frac{im\,(x_{n+1}-x)^2}{2\hbar(n+1)\Delta t}} \quad \checkmark$$

confirms the inductive Ansatz (1).
Using (4) and setting $n=N$, and combining with the overall prefactor $\left(\frac{m}{ih\Delta t}\right)^{N/2}$ gives the final result (over)

$$\langle x', t' | x, t \rangle = \sqrt{\frac{m}{i\hbar N\Delta t}} \, e^{\frac{im(x'-x)^2}{2\hbar N\Delta t}},$$

with $N\Delta t = t' - t$, we see that the result is independent of N.

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2) a) Let us shift the trajectory $x(t'')$ by a small amount $\delta x(t'')$ with the b.c. that $\delta x(t) = 0$ and $\delta x(t') = \delta x'$. Under this variation,

$$\langle x' + \delta x', t' | x, t \rangle - \langle x', t' | x, t \rangle = \frac{\partial}{\partial x'} \langle x', t' | x, t \rangle \, \delta x(t').$$

The path integral changes by

$$\int \mathcal{D}[x(t)] \, e^{i \frac{S[x(t) + \delta x(t)]}{\hbar}} - \int \mathcal{D}[x(t)] \, e^{i \frac{S[x(t)]}{\hbar}}$$

$$= \int \mathcal{D}[x(t)] \, i \frac{\delta S}{\hbar} \, e^{i \frac{S[x(t)]}{\hbar}}$$

$$\delta S = S[x(t) + \delta x(t)] - S[x(t)]$$

$$= \int_t^{t'} dt'' \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right)$$

$$\delta S = \frac{\partial L}{\partial \dot{x}} \delta x \Big|_x^{x'} + \int_t^{t'} dt'' \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x$$

$= p(t') \delta x(t')$. The second term of zero classically, and we shall see that its QM average is also zero.

Then

$$\frac{\partial}{\partial x'} \langle x', t' | x, t \rangle = \int \mathcal{D}[x(t)] \, i \frac{P_x(t')}{\hbar} e^{i \frac{S[x(t)]}{\hbar}} .$$

Now, to show that the Euler-Lagrange equation is obeyed in an average sense. Consider $x(t'') \to x(t'') + \delta x(t'')$, with $\delta x(t) = 0 = \delta x(t')$. This doesn't change the path integral:

$$0 = \int \mathcal{D}[x(t)] \, e^{i \frac{S[x(t) + \delta x(t)]}{\hbar}} - \int \mathcal{D}[x(t)] \, e^{i \frac{S[x(t)]}{\hbar}}$$

$$= \int \mathcal{D}[x(t)] \, i \frac{\delta S}{\hbar} e^{i \frac{S[x(t)]}{\hbar}}$$

$$\delta S = \int \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x(t)\, dt \, .$$

$$\therefore \int \mathcal{D}[x(t)] \left(\frac{i}{\hbar} \int \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x(t)\, dt \right) e^{\frac{i S[x(t)]}{\hbar}} = 0$$

Because $\delta x(t)$ is arbitrary (except for b.c.'s), the expression must be zero at all t independently,

$$\int \mathcal{D}[x(t)] \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) e^{\frac{i S[x(t)]}{\hbar}} = 0 .$$

Euler-Lagrange equation holds as a QM expectation value (Feynman version of Ehrenfest theorem).

b) $$\frac{\partial}{\partial t'} \langle x', t' | x, t \rangle = \int \mathcal{D}[x(t)] \, \frac{i}{\hbar} \frac{\partial S}{\partial t'} \, e^{\frac{i S[x(t)]}{\hbar}}$$

But $\frac{\partial S}{\partial t'} = -H(t')$ (see e.g., Goldstein).

$$\therefore \frac{\partial}{\partial t'} \langle x', t' | x, t \rangle = \int \mathcal{D}[x(t)] \left(-\frac{i H(t')}{\hbar} \right) e^{\frac{i S[x(t)]}{\hbar}}$$

c) If $H = \frac{p^2}{2m} + V(x)$, then

$$\int \mathcal{D}[x(t)] \left(H - \frac{p^2}{2m} - V(x)\right) e^{i\frac{S[x(t)]}{\hbar}} = 0$$

$$i\hbar \frac{\partial}{\partial t'} \langle x', t' | x, t \rangle = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x'^2} + V(x')\right] \langle x', t' | x, t \rangle,$$

which is just Schrödinger's equation.