Exercises for Physics 570A

Problem Set 9; Due in class Tuesday, November 26

1) A particle is known to be in an eigenstate of $\vec{L}^2$ and L_z with eigenvalues $\hbar^2\ell(\ell+1)$ and $m\hbar$, respectively. Prove the following expectation values:

$$\langle L_x \rangle = \langle L_y \rangle = 0,$$

$$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m^2\right].$$

2) Show that the orbital angular momentum operator $\vec{L}$ commutes with both the operators $\vec{r}^2$ and $\vec{p}^2$.

3) Suppose half-odd integer values, say $\ell = 1/2$, were allowed for orbital angular momentum. From

$$L_+ Y_{1/2,1/2}(\theta,\phi) = 0,$$

we may deduce

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}.$$

Now try to construct $Y_{1/2,-1/2}(\theta,\phi)$ by (a) applying L_- to $Y_{1/2,1/2}(\theta,\phi)$; and (b) using $L_- Y_{1/2,-1/2}(\theta,\phi) = 0$. Show that the two procedures lead to contradictory results, refuting the notion of half-odd integer orbital quantum numbers.

4) (a) Construct the matrix representation of $\hat{S}_r = \vec{S}\cdot\hat{r}$, representing the component of spin angular momentum along an arbitrary direction

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}.$$

(b) Find the eigenvalues and (normalized) eigenvectors of $\hat{S}_r$.

5) Find the linear combination $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ that maximizes the uncertainty product

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle.$$

Verify that for the linear combination you found, the uncertaintly relation for S_x and S_y is satisfied.

6) A beam of spin-1/2 atoms goes through a series of Stern-Gerlach-type measurements as follows:
(a) The first measurement accepts $S_z = +\hbar/2$ atoms and rejects $S_z = -\hbar/2$ atoms.
(b) The second measurement accepts accepts $S_r = +\hbar/2$ atoms and rejects $S_r = -\hbar/2$ atoms, where S_r is defined in problem 4.
(c) The third measurement accepts $S_z = -\hbar/2$ atoms and rejects $S_z = +\hbar/2$ atoms.
What is the intensity of the final $S_z = -\hbar/2$ beam if the $S_z = +\hbar/2$ beam after the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $S_z = -\hbar/2$ beam?

7) Zeemann precession

Consider a spin-1/2 particle with magnetic moment operator $\vec{\mu} = \mu_B \vec{\sigma}$, where $\mu_B = e\hbar/2m$ is the Bohr magneton and $\vec{\sigma}$ is a vector of Pauli matrices. The particle is placed in a constant magnetic field $\vec{B} = B\hat{z}$, with Hamiltonian $\hat{H} = -\vec{\mu} \cdot \vec{B}$.

(a) Write the Heisenberg equations of motion for the operators $\hat{S}_x(t)$, $\hat{S}_y(t)$, and $\hat{S}_z(t)$.

(b) Solve them to obtain $\vec{S}(t)$.