Phys 570A HW 9 Solutions

4) $\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\,\vec{\sigma}\cdot\hat{n}$

$$\vec{\sigma}\cdot\hat{n} = n_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + n_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + n_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$\vec{\sigma}\cdot\hat{n}\,|+\rangle = |+\rangle$$

$$(\vec{\sigma}\cdot\hat{n} - \mathbb{1})|+\rangle = 0$$

$$\begin{pmatrix} n_z - 1 & n_x - i n_y \\ n_x + i n_y & -n_z - 1 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(n_z - 1)\alpha + (n_x - i n_y)\beta = 0$$

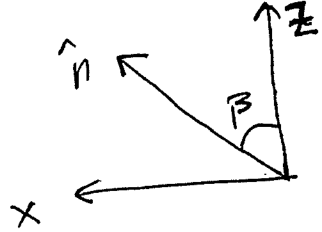
$$\beta = \frac{1-n_z}{n_x - i n_y}\alpha, \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$|\alpha|^2\left(1 + \frac{(1-n_z)^2}{n_x^2+n_y^2}\right) = 1$$

$$|\alpha|^2 = \frac{1\,(n_x^2+n_y^2)}{n_x^2+n_y^2+n_z^2-2n_z+1} = \frac{n_x^2+n_y^2}{2(1-n_z)}$$

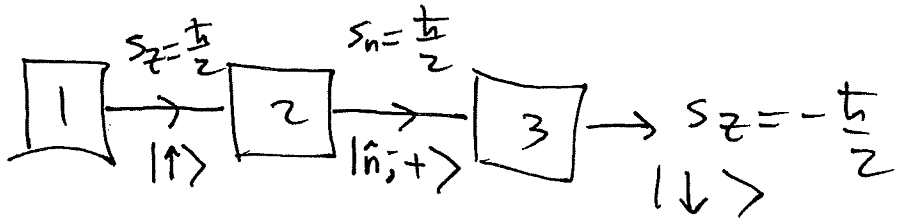
Let $\alpha = \dfrac{n_x - i n_y}{\sqrt{2(1-n_z)}}$ then $\beta = \sqrt{\dfrac{1-n_z}{2}}$

$$|\vec{S}\cdot\hat{n};+\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

6) 

$n_x = \sin\beta$ $\quad n_z = \cos\beta$

$n_y = 0$



$$\alpha = \frac{\sin\beta}{\sqrt{2(1-\cos\beta)}}, \quad \beta = \sqrt{\frac{1-\cos\beta}{2}} = \sin\frac{\beta}{2}$$

$$\alpha = \cos\frac{\beta}{2}$$

$$|S_n = \tfrac{\hbar}{2}\rangle = \cos\frac{\beta}{2}|\uparrow\rangle + \sin\frac{\beta}{2}|\downarrow\rangle$$

$$|\langle S_n = \tfrac{\hbar}{2}|\uparrow\rangle|^2 = \cos^2\frac{\beta}{2}$$

$$\cancel{A}\downarrow|S_n = \tfrac{\hbar}{2}\rangle|^2 = \sin^2\frac{\beta}{2}$$

$$I = \cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta, \quad \text{max @ } \beta = \frac{\pi}{2}$$

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5) Let $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

$$\langle S_x\rangle = \frac{\hbar}{2}(\alpha^* \ \beta^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\hbar}{2}(\alpha^*\beta + \alpha\beta^*)$$

$$\langle S_y\rangle = \frac{\hbar}{2}(\alpha^* \ \beta^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\hbar}{2}i(\alpha\beta^* - \alpha^*\beta)$$

Let $\alpha = \cos\frac{\theta}{2}$, $\beta = e^{i\phi}\sin\frac{\theta}{2}$

$$\langle S_x \rangle = \frac{\hbar}{2}\left(e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{\hbar}{2}\sin\theta\cos\phi$$

$$\langle S_y \rangle = \frac{\hbar}{2} i\left(e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - e^{+i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{\hbar}{2}\sin\theta\sin\phi$$

$$(\Delta S_x)^2 = \frac{\hbar^2}{4}\left(1 - \sin^2\theta\cos^2\phi\right)$$

$$(\Delta S_y)^2 = \frac{\hbar^2}{4}\left(1 - \sin^2\theta\sin^2\phi\right)$$

$$(\Delta S_x)^2(\Delta S_y)^2 = \left(\frac{\hbar}{2}\right)^4\left(1 - \sin^2\theta\cos^2\phi\right)\left(1 - \sin^2\theta\sin^2\phi\right)$$

$$= \left(\frac{\hbar}{2}\right)^4\left(1 - \sin^2\theta + \sin^4\theta\,\frac{\sin^2 2\phi}{4}\right)$$

This is maximized for $\sin^2 2\phi = 1$. Then

$(\Delta S_x)^2(\Delta S_y)^2 = \left(\frac{\hbar}{2}\right)^4\left(1 - \sin^2\theta + \frac{\sin^4\theta}{4}\right)$. max

when $\sin\theta = 0$. $(\Delta S_x)^2(\Delta S_y)^2 \leq \left(\frac{\hbar}{2}\right)^4$

$\Delta S_x \, \Delta S_y \geq \frac{\hbar}{2}|\langle S_z \rangle|$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(|\alpha|^2 - |\beta|^2\right)$$
$$= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = \frac{\hbar}{2}\cos\theta$$

Saturates uncertainty bound.

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7) $$\frac{d\vec{S}}{dt} = \frac{1}{i\hbar}[\vec{S}, H] = \frac{\omega}{i\hbar}[\vec{S}, S_z]$$

$$\Rightarrow \frac{dS_z}{dt} = 0 \qquad \frac{dS_x}{dt} = \frac{\omega}{i\hbar}[S_x, S_z] = -\omega S_y$$

$$\frac{dS_y}{dt} = \frac{\omega}{i\hbar}[S_y, S_z] = \omega S_x$$

$$\omega = -\frac{eB}{m}$$

$$\frac{d^2 S_x}{dt^2} = -\omega \frac{dS_y}{dt} = -\omega^2 S_x$$

$$S_x(t) = A\cos(\omega t + \delta)$$
$$S_y(t) = -\frac{1}{\omega}\frac{dS_x}{dt} = A\sin(\omega t + \delta)$$
$$S_z = \text{const.}$$

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1) $$L_\pm = L_x \pm i L_y$$

$$L_x = \frac{L_+ + L_-}{2}, \qquad L_y = \frac{L_+ - L_-}{2i}$$

$$\langle lm | L_\pm | lm \rangle = 0 \quad \therefore \quad \langle L_x \rangle = \langle L_y \rangle = 0$$

$$L_x^2 = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right)$$

$$L_\pm |\ell m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\,|\ell\, m\pm 1\rangle$$

$$\langle L_x^2\rangle = \frac{1}{4}\langle \ell m|\, L_+L_- + L_-L_+\,|\ell m\rangle$$

$$L_-L_+|\ell m\rangle = \hbar^2\sqrt{\ell(\ell+1)-(m+1)m}\sqrt{\ell(\ell+1)-m(m+1)}\,|\ell m\rangle$$

$$= \hbar^2\left(\ell(\ell+1) - m(m+1)\right)|\ell m\rangle$$

$$L_+L_-|\ell m\rangle = \hbar^2\sqrt{\ell(\ell+1)-(m-1)m}\sqrt{\ell(\ell+1)-m(m-1)}\,|\ell m\rangle$$

$$= \hbar^2\left(\ell(\ell+1) - m(m-1)\right)|\ell m\rangle$$

$$\langle L_x^2\rangle = \frac{\hbar^2}{2}\left(\ell(\ell+1) - m^2\right)$$

$$L_y^2 = \frac{1}{4}\left(-L_+^2 - L_-^2 + L_+L_- + L_-L_+\right)$$

$$\Rightarrow \langle L_y^2\rangle = \langle L_x^2\rangle.$$

For semiclassical interpretation, see discussion of "vector model" in Lec. 23 (at the end).

3) Suppose $Y_{1/2\, 1/2}(\theta,\phi) = C e^{i\phi/2}\sqrt{\sin\theta}$

$$L_- = +i\hbar e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_- Y_{1/2\,1/2} = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\; Y_{\frac{1}{2},-1/2} = \hbar\, Y_{\frac{1}{2},-\frac{1}{2}}$$

$$L_- Y_{\frac{1}{2}\frac{1}{2}} = i\hbar e^{-i\phi} C\left(i\frac{\partial}{\partial\theta}\sqrt{\sin\theta}\, e^{i\phi/2} + \cot\theta\frac{\partial}{\partial\phi} e^{i\phi/2}\sqrt{\sin\theta}\right)$$

$$= i\hbar C e^{-i\phi/2}\left(\frac{i\cos\theta}{2\sqrt{\sin\theta}} + \frac{\cos\theta}{\sqrt{\sin\theta}}\frac{i}{2}\right)$$

$$= -\hbar C e^{-i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}$$

$$\Rightarrow Y_{\frac{1}{2},-\frac{1}{2}} = -C e^{-i\phi/2}\cot\theta\sqrt{\sin\theta}$$

But we must also have

$$0 = L_- Y_{\frac{1}{2},-\frac{1}{2}} = i\hbar e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right) e^{-i\frac{\phi}{2}}\tilde{C} f(\theta)$$

$$0 = i f'(\theta) + \cot\theta\left(-\tfrac{i}{2}\right) f(\theta)$$

$$f'(\theta) = \tfrac{1}{2}\cot\theta\, f(\theta)$$

$$\frac{df}{f} = \frac{1}{2}\frac{d\sin\theta}{\sin\theta} \qquad \ln f = \tfrac{1}{2}\ln\sin\theta + B$$

$$f(\theta) = \tilde{C}\sqrt{\sin\theta}$$

$$Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) = \tilde{c}\, e^{-i\frac{\phi}{2}}\sqrt{\sin\theta}$$

The two expressions are inconsistent.

$\Rightarrow$ Contradiction! Half-odd integer values are not allowed for orbital angular momentum.

2) $\vec{r}^2$ and $\vec{p}^2$ are invariant under rotations. An infinitessimal rotation about the axis $\hat{n}$ by the angle $\delta\theta$ is described by the transformation

$$\hat{U} = e^{\frac{i\,\delta\theta\,\hat{n}\cdot\vec{L}}{\hbar}} \simeq \mathbb{1} + \frac{i}{\hbar}\delta\theta\,\hat{n}\cdot\vec{L}$$

$$\vec{r}\,' = \hat{U}\,\vec{r}\,\hat{U}^{+}$$

$$(\vec{r}\,')^2 = \hat{U}\,\vec{r}^{\,2}\,\hat{U}^{+} = \left(\mathbb{1} + i\frac{\delta\theta}{\hbar}\hat{n}\cdot\vec{L}\right)\vec{r}^{\,2}\left(\mathbb{1} - i\frac{\delta\theta}{\hbar}\hat{n}\cdot\vec{L}\right)$$

$$\vec{r}^{\,2} = \vec{r}^{\,2} + \frac{i\delta\theta}{\hbar}\left[\hat{n}\cdot\vec{L}, \vec{r}^{\,2}\right] + \mathcal{O}(\delta\theta^2)$$

$$\Rightarrow \left[\hat{n}\cdot\vec{L}, \vec{r}^{\,2}\right] = 0$$

$$\hat{n}\cdot\left[\vec{L}, \vec{r}^{\,2}\right] = 0 \qquad \forall\ \hat{n}$$

$$\therefore \left[\vec{L}, \vec{r}^{\,2}\right] = 0$$

Similarly, $(\vec{p}\,')^2 = \vec{p}^{\,2}$, so

$$\left[\vec{L}, \vec{p}^{\,2}\right] = 0.$$