

1) The inner product $\langle \phi | \psi \rangle$

Let \hat{Q} be a Hermitian operator,
 with $\hat{Q} \psi_n = \delta_n \psi_n$. From the
 completeness of $\{\psi_n\}$, we may

write
$$\psi(x) = \sum_n c_n \psi_n(x),$$

$$\phi(x) = \sum_n d_n \psi_n(x).$$

kets: $|\psi\rangle \leftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} \quad |\phi\rangle \leftrightarrow \begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix}$

$$\langle \phi | \psi \rangle = \sum_n \langle \phi | n \rangle \langle n | \psi \rangle$$

$$= \sum_n d_n^* c_n = (d_1^* \ d_2^* \ \dots) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

Thus the inner product of two quantum states is the scalar product of the corresponding vectors of coefficients, but where the left vector is complex conjugated:

$$\langle \psi | \leftrightarrow (c_1^* \ c_2^* \ \dots)$$

bras:

$$\langle \phi | \leftrightarrow (d_1^* \ d_2^* \ \dots)$$

Kets are equivalent to column vectors, while bras are equivalent to complex conjugated row vectors.

The standard definition

$$\langle \phi | \psi \rangle = \int dx \phi^*(x) \psi(x) = \int dx \langle \phi | x \rangle \langle x | \psi \rangle$$

is just the expression in the position basis.

2) Operators in bra-ket notation

When an operator \hat{Q} acts on a state vector $|\psi\rangle$, it creates another (not necessarily normalized) state vector $|\phi\rangle$:

$$|\phi\rangle = \hat{Q}|\psi\rangle.$$

Taking the components of this equation in an orthonormal basis $|n\rangle$, $n=1, 2, \dots$, gives

$$\langle n|\phi\rangle = \langle n|\hat{Q}|\psi\rangle = \sum_{n'} \langle n|\hat{Q}|n'\rangle \langle n'|\psi\rangle$$

Using $d_n = \langle n|\phi\rangle$, $c_n = \langle n|\psi\rangle$,

and $Q_{nn'} = \langle n|\hat{Q}|n'\rangle$, this becomes

$$d_n = \sum_{n'} Q_{nn'} c_{n'}. \quad \text{Or}$$

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & \dots \\ Q_{21} & Q_{22} & \\ \vdots & & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

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The action of an operator on a state vector is completely analogous to the action of a matrix on a column vector.

If \hat{Q} is Hermitian, then

$$\hat{Q}^\dagger = \hat{Q} \quad \text{or} \quad Q_{ij}^* = Q_{ji}$$

\Rightarrow diagonal elements are real.

The corresponding bra equation is

$$\langle \phi | = \langle \psi | \hat{Q}^\dagger$$

$$\langle \phi | n \rangle = \langle \psi | \hat{Q}^\dagger | n \rangle = \sum_{n'} \langle \psi | n' \rangle \langle n' | \hat{Q}^\dagger | n \rangle$$

$$d_n^* = \sum_{n'} c_{n'}^* (\hat{Q}^\dagger)_{n'n} \quad \text{or}$$

$$(d_1^* \ d_2^* \ \dots) = (c_1^* \ c_2^* \ \dots) \begin{pmatrix} Q_{11}^+ & Q_{12}^+ & \dots \\ Q_{21}^+ & Q_{22}^+ & \\ \vdots & & \end{pmatrix} \quad (5)$$

If \hat{Q} is Hermitian, then $\hat{Q}^\dagger = \hat{Q}$, i.e., the matrix which acts to the left on row vectors is the same matrix which acts to the right on column vectors.

3) Operators in the $|x\rangle$ basis

$$|\phi\rangle = \hat{Q} |\psi\rangle$$

$$\langle x | \phi \rangle = \langle x | \hat{Q} | \psi \rangle = \int dx' \langle x | \hat{Q} | x' \rangle \langle x' | \psi \rangle$$

$$\phi(x) = \int dx' Q(x, x') \psi(x')$$

(c) Position operator

$$\begin{aligned} \langle x | \hat{x} | x' \rangle &= x' \langle x | x' \rangle \\ &= x' \delta(x - x') \\ &= x \delta(x - x') \end{aligned}$$

Formally, we can write

$$\hat{X} = \int_{-\infty}^{\infty} dx' x' |x'\rangle \langle x'|$$

$$\hat{X} |\psi\rangle = \int_{-\infty}^{\infty} dx' x' |x'\rangle \langle x' | \psi \rangle$$

$$= \int_{-\infty}^{\infty} dx' |x'\rangle x' \psi(x')$$

$$\langle x | \hat{X} | \psi \rangle = \int_{-\infty}^{\infty} dx' \underbrace{\langle x | x' \rangle}_{\delta(x-x')} x' \psi(x')$$

$$= x \psi(x)$$

ii) Momentum operator

$$\langle \phi | \hat{p}_x | \psi \rangle = \int_{-\infty}^{\infty} dx \phi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x)$$

$$= \int_{-\infty}^{\infty} dx \langle \phi | x \rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | \psi \rangle$$

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$$= \langle \phi | \left(\int_{-\infty}^{\infty} dx |x\rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x| \right) | \psi \rangle$$

$$\Rightarrow \hat{P}_x = \int_{-\infty}^{\infty} dx |x\rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|$$

Similarly,

$$\hat{H} = \int_{-\infty}^{\infty} dx |x\rangle \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \langle x|$$

4) Operators in the $|k\rangle$ basis

$$\langle k | \phi \rangle = \langle k | \hat{Q} | \psi \rangle$$

$$\langle k | \phi \rangle = \int \frac{dk'}{2\pi} \langle k | \hat{Q} | k' \rangle \langle k' | \psi \rangle$$

$$\tilde{\phi}(k) = \int \frac{dk'}{2\pi} Q(k, k') \tilde{\psi}(k')$$

$$\hat{Q} = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle Q(k, k') \langle k'|$$

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i) Momentum operator

$$\langle k | \hat{P}_x | k' \rangle = \hbar k' \langle k | k' \rangle = \hbar k' 2\pi \delta(k - k')$$

$$\hat{P}_x = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle \hbar k 2\pi \delta(k - k') \langle k'|$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hbar k |k\rangle \langle k|$$

In three dimensions, one would have

$$\vec{P} = \int \frac{d^3k}{(2\pi)^3} \hbar \vec{k} |\vec{k}\rangle \langle \vec{k}|$$

ii) Potential energy

$$\langle k | V | k' \rangle = \int_{-\infty}^{\infty} dx e^{-ikx} V(x) e^{ik'x}$$

$$V(k, k') = \int_{-\infty}^{\infty} dx V(x) e^{i(k' - k)x}$$

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$$\equiv \tilde{V}(k - k')$$

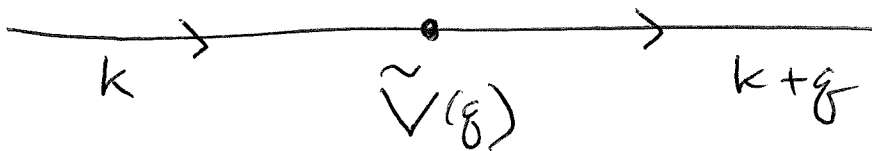
Fourier
transform

$$\hat{V} = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle V(k, k') \langle k'|$$

$$= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \tilde{V}(k - k') |k\rangle \langle k'|$$

$$= \int \frac{dq}{2\pi} \int \frac{dk}{2\pi} \tilde{V}(q) |k+q\rangle \langle k|$$

Feynman diagram:



$$\Psi(x,t) = \langle x | \Psi(t) \rangle$$

Schrödinger eq. for $|\Psi\rangle$:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = i\hbar \frac{\partial}{\partial t} \langle x | \Psi(t) \rangle$$

$$= \langle x | i\hbar \frac{\partial}{\partial t} | \Psi(t) \rangle$$

$$\frac{\partial}{\partial t} \langle x | = 0 \quad (\text{fixed basis vector})$$

$$\hat{H} \Psi(x,t) = \langle x | \hat{H} | \Psi(t) \rangle$$

$$i\hbar \frac{\partial}{\partial t} | \Psi(t) \rangle = \hat{H} | \Psi(t) \rangle$$

If \hat{H} is independent of time,
the formal solution is

$$| \Psi(t) \rangle = \hat{U}(t) | \Psi(0) \rangle$$

$$\hat{U}(t) = \exp\left(-i \frac{\hat{H} t}{\hbar}\right)$$