# Physics 570A Lecture 12

## 1) Measurements of compatible observables: Commuting operators

Recall $[\hat{Q}, \hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q}$.

If $[\hat{Q}, \hat{P}] = 0$, then

$\langle \hat{Q}\hat{P} \rangle = \langle \hat{P}\hat{Q} \rangle$ i.e., the result does not depend on the order of measurement

Theorem If $[\hat{P}, \hat{Q}] = 0$ and either $\hat{P}$ or $\hat{Q}$ has non-degenerate eigenvalues, its eigenfunctions are also eigenfunctions of the other operator.

Proof: Given $[\hat{P}, \hat{Q}] = 0$ and

$\hat{P}\psi_i = p_i \psi_i$, where all $p_i$ are distinct.

Then, $\hat{Q}\hat{P}\psi_i = \hat{Q}\, p_i \psi_i = p_i(\hat{Q}\psi_i)$.

But $\hat{Q}\hat{P}\psi_i = \hat{P}\hat{Q}\psi_i = \hat{P}(\hat{Q}\psi_i)$

$$\Rightarrow \hat{P}(\hat{Q}\psi_i) = p_i(\hat{Q}\psi_i).$$

$\hat{Q}\psi_i$ is an eigenvector of $\hat{P}$ with eigenvalue $p_i$. This leads to a contradiction unless

$\hat{Q}\psi_i = q_i \psi_i$, where $q_i$ is a complex number. Therefore $\psi_i$ is also an eigenfunction of $\hat{Q}$.

Let us write
$\psi = \sum_i c_i \psi_i$. If the variable $p$ is measured, the result $p_i$ will be obtained with probability $|c_i|^2$, assuming $\psi$ is normalized. After the measurement, the wavefunction will be $\psi_i$. Subsequent measurements of $q$ or $p$ will yield $q_i$ and $p_i$, respectively, and $\psi$ will not be altered by further measurements of these variables. $q$ and $p$ are said to be compatible.

Q: What happens if $[\hat{Q}, \hat{P}] \neq 0$? (4

2) Commutators and uncertainty relations

If $[\hat{Q}, \hat{P}] \neq 0$, it is clear that $\hat{Q}$ and $\hat{P}$ do not have the same eigenfunctions.

Measurement of $q$ forces $\psi$ into an eigenfunction of $\hat{Q}$.

A subsequent measurement of $\hat{P}$ forces $\psi$ into an eigenfunction of $\hat{P}$, and destroys the information about the variable $q$ gleaned from the previous measurement.

This is the root of the

# Uncertainty principle.

## Generalized uncertainty principle

$$\Delta Q \, \Delta P \geq \frac{1}{2} \left| \langle [\hat{Q}, \hat{P}] \rangle \right| ,$$

where
$$(\Delta Q)^2 = \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2$$
$$(\Delta P)^2 = \langle \hat{P}^2 \rangle - \langle \hat{P} \rangle^2 .$$

Proof: Let $\Delta \hat{Q} = \hat{Q} - \langle \psi | \hat{Q} | \psi \rangle$ for a given wavefunction $\psi$. Similarly, define $\Delta \hat{P} = \hat{P} - \langle \psi | \hat{P} | \psi \rangle$.

$$\langle (\Delta \hat{Q})^2 \rangle = \langle \hat{Q}^2 \rangle + \langle \hat{Q} \rangle^2 - 2 \langle \hat{Q} \rangle^2$$
$$= \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2 = (\Delta Q)^2$$

Also $\langle (\Delta \hat{P})^2 \rangle = (\Delta P)^2$. 

Now $(\Delta Q)^2 (\Delta P)^2 = \langle \Psi | (\Delta \hat{Q})^2 | \Psi \rangle$
$\times \langle \Psi | (\Delta \hat{P})^2 | \Psi \rangle$

$= \langle \Delta \hat{Q} \Psi | \Delta \hat{Q} \Psi \rangle \langle \Delta \hat{P} \Psi | \Delta \hat{P} \Psi \rangle$,

since $\Delta \hat{Q}$ and $\Delta \hat{P}$ are hermitian.

$(\Delta Q)^2 (\Delta P)^2 \geq | \langle \Delta \hat{Q} \Psi | \Delta \hat{P} \Psi \rangle |^2$

(schwartz inequality)

$\geq \left[ \mathrm{Im} \left( \langle \Delta \hat{Q} \Psi | \Delta \hat{P} \Psi \rangle \right) \right]^2$

$= \left| \dfrac{\langle \Delta \hat{Q} \Psi | \Delta \hat{P} \Psi \rangle - \langle \Delta \hat{P} \Psi | \Delta \hat{Q} \Psi \rangle}{2i} \right|^2$

$= \left| \langle [\Delta \hat{Q}, \Delta \hat{P}] \rangle / 2 \right|^2$

$= \left| \langle [\hat{Q}, \hat{P}] \rangle / 2 \right|^2$

$$\Rightarrow \Delta Q \, \Delta P \geq \frac{1}{2} \left| \langle [\hat{Q}, \hat{P}] \rangle \right| .$$

Example $\quad \hat{x} = x \quad \hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$

$$[\hat{x}, \hat{p}_x] \psi = \frac{\hbar}{i} \left( x \frac{\partial}{\partial x} \psi - \frac{\partial}{\partial x} (x \psi) \right)$$

$$= \frac{\hbar}{i} \left[ x \psi'(x) - \psi(x) - x \psi'(x) \right]$$

$$= i\hbar \, \psi(x)$$

$$\Rightarrow [\hat{x}, \hat{p}_x] = i\hbar$$

$$\Delta x \, \Delta p_x \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{p}_x] \rangle \right| = \frac{\hbar}{2}$$

The familiar form of the uncertainty principle.

Aside: The Schwartz inequality for two vectors states

$$|\vec{a}|\,|\vec{b}| \geq |\vec{a}\cdot\vec{b}|.$$

In terms of two complex functions $f(x)$ and $g(x)$, one has analogously

$$\int |f(x)|^2 dx \int |g(x)|^2 dx \geq \left| \int f^*(x)\, g(x)\, dx \right|^2.$$

## 3) Time evolution of expectation values

$$\frac{d}{dt}\langle \hat{Q} \rangle = \frac{d}{dt}\langle \Psi | \hat{Q} \Psi \rangle$$

$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q} \Psi \right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t} \Psi \right\rangle + \left\langle \Psi \middle| \hat{Q} \frac{\partial \Psi}{\partial t} \right\rangle$$

write it out long-hand

But $\dfrac{\partial \Psi}{\partial t} = \dfrac{\hat{H}\Psi}{i\hbar}$ and

$$\frac{\partial \Psi^*}{\partial t} = -\frac{\hat{H}\Psi^*}{i\hbar},$$ so

$$\frac{d\langle \hat{Q}\rangle}{dt} = -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle$$

$$\frac{d}{dt}\langle \hat{Q}\rangle = -\frac{1}{i\hbar}\langle \Psi | (\hat{H}\hat{Q} - \hat{Q}\hat{H})\Psi\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$= -\frac{1}{i\hbar}\langle [\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$= \frac{1}{i\hbar}\langle [\hat{Q},\hat{H}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

Thus the expectation value is constant if $\hat{Q}$ has no explicit time dependence and it commutes with the Hamiltonian.

**Example**

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$= \frac{\hat{P}_x^2}{2m} + V(x)$$

i) $$\frac{d}{dt}\langle x\rangle = -\frac{1}{i\hbar}\langle[\hat{H},\hat{x}]\rangle$$

$$= -\frac{1}{2i\hbar m}\langle[\hat{P}_x^2,\hat{x}]\rangle$$

$$= -\frac{1}{2i\hbar m}\langle \hat{P}_x[\hat{P}_x,\hat{x}] + [\hat{P}_x,\hat{x}]\hat{P}_x\rangle$$

$$= \frac{1}{m}\langle\hat{P}_x\rangle$$

$$m\frac{d}{dt}\langle x\rangle = \langle\hat{P}_x\rangle$$

ii) $$\frac{d}{dt}\langle P_x\rangle = \frac{1}{i\hbar}\langle[\hat{P}_x,\hat{H}]\rangle$$

$$= \frac{1}{i\hbar}\langle[\hat{P}_x,V(x)]\rangle$$

$$[\hat{p}_x, V(x)]\psi(x) = \frac{\hbar}{i}\frac{d}{dx}\big(V(x)\psi(x)\big) - V(x)\frac{\hbar}{i}\frac{d\psi}{dx}$$

$$= \frac{\hbar}{i} V'(x)\,\psi(x)$$

$$\Rightarrow \quad [\hat{p}_x, \hat{V}(x)] = \frac{\hbar}{i}\frac{dV}{dx}$$

$$\frac{d}{dt}\langle p_x \rangle = -\left\langle \frac{dV}{dx} \right\rangle .$$

Thus the expectation values obey the classical equations of motion! This result is known as Ehrenfest's theorem, and holds quite generally.

## Addendum

$$(\Delta Q)^2 (\Delta P)^2 \geq \left| \langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle \right|^2$$

$$= \left[ \mathrm{Im} \langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle \right]^2 + \left[ \mathrm{Re} \langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle \right]^2$$

$$= \left[ \frac{\langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle - \langle \Delta\hat{P}\psi | \Delta\hat{Q}\psi \rangle}{2i} \right]^2$$

$$+ \left[ \frac{\langle \Delta\hat{Q}\psi | \Delta\hat{P}\psi \rangle + \langle \Delta\hat{P}\psi | \Delta\hat{Q}\psi \rangle}{2} \right]^2$$

$$= \left| \langle [\hat{Q}, \hat{P}] \rangle / 2i \right|^2 + \left| \langle \{ \Delta\hat{Q}, \Delta\hat{P} \} \rangle / 2 \right|^2$$

$$\langle \{ \Delta\hat{Q}, \Delta\hat{P} \} \rangle = \langle \{ \hat{Q} - \langle \hat{Q} \rangle, \hat{P} - \langle \hat{P} \rangle \} \rangle$$

$$= \langle \{ \hat{Q}, \hat{P} \} \rangle + 2 \langle \hat{Q} \rangle \langle \hat{P} \rangle$$

$$- \langle \{ \langle \hat{Q} \rangle, \hat{P} \} \rangle - \langle \{ \hat{Q}, \langle \hat{P} \rangle \} \rangle$$

$$= \langle \{ \hat{Q}, \hat{P} \} \rangle - 2 \langle \hat{Q} \rangle \langle \hat{P} \rangle$$

So a more stringent lower bound on uncertainty is

$$\boxed{\begin{aligned}(\Delta Q)^2(\Delta P)^2 &\geq \left|\frac{\langle[\hat{Q},\hat{P}]\rangle}{2}\right|^2 \\ &\quad + \left|\frac{1}{2}\langle\{\hat{Q},\hat{P}\}\rangle - \langle\hat{Q}\rangle\langle\hat{P}\rangle\right|^2\end{aligned}}$$

$$\geq \left|\frac{\langle \hat{Q},\hat{P}\rangle}{2}\right|^2$$ (usual uncertainty bound)

In many cases,

$$\frac{1}{2}\langle\{\hat{Q},\hat{P}\}\rangle - \langle\hat{Q}\rangle\langle\hat{P}\rangle = \left\langle\frac{\hat{Q}\hat{P}+\hat{P}\hat{Q}}{2}\right\rangle - \langle\hat{Q}\rangle\langle\hat{P}\rangle$$

is simply zero, so the extra term doesn't change anything.