Physics 570A

Parity

Define the <u>parity operator</u> $\hat{P}$:

$$\hat{P} f(x) = f(-x)$$

$$\hat{P}^2 f(x) = \hat{P} f(-x) = f(x)$$

$$\Rightarrow \hat{P}^2 = \mathbb{1} \quad \text{unit operator}$$

<u>Eigenfunctions & eigenvalues</u>:

Let $\hat{P}\,\psi_p(x) = p\,\psi_p(x)$

$$\hat{P}^2 \psi_p(x) = p^2 \psi_p(x) = \psi_p(x)$$

$$\Rightarrow p^2 = 1, \quad p = \pm 1$$

i) $p = +1$ $\quad \psi_p(-x) = \psi_p(x)$ $\quad$ (even function)

ii) $p = -1$ $\quad \psi_p(-x) = -\psi_p(x)$ $\quad$ (odd function)

$\Rightarrow$ The eigenfunctions of the parity operator are the even and odd functions.

[The same holds true for wavefunctions in three dimensions:

$$\hat{P}\psi(\vec{x}) = \psi(-\vec{x}), \text{ etc.}$$
]

Even potentials $\quad V(-x) = V(x)$

Consider

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x).$$

an energy eigenstate:

$$\hat{H}\psi(x) = E\,\psi(x)$$

$$\hat{P}\hat{H}\psi(x) = E\,\hat{P}\psi(x) = E\,\psi(-x)$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{d(-x)^2} + V(-x)\right)\psi(-x) = E\,\psi(-x)$$

$$\hat{H}\,\psi(-x) = E\,\psi(-x).$$

$\Rightarrow$ $\psi(-x)$ is an eigenstate of $\hat{H}$ with the same energy eigenvalue as $\psi(x)$! If the energy E is nondegenerate, then

$$\psi(-x) = e^{i\theta}\,\psi(x). \quad \text{That is,}$$

$$\hat{P}\,\psi(x) = e^{i\theta}\,\psi(x).$$

But $\hat{P}^2\,\psi(x) = e^{i\theta}\,\hat{P}\,\psi(x) = e^{i2\theta}\,\psi(x)$

$$\psi(x) \overset{\shortparallel}{} = e^{i2\theta}\,\psi(x)$$

$$\Rightarrow e^{i2\theta} = 1, \quad e^{i\theta} = p = \pm 1$$

$$\boxed{\text{Thus the energy eigenstates in a symmetric potential are also eigenstates of parity.}}$$

Translation operator

$$\hat{T}(\vec{a})\,\psi(\vec{x}) = \psi(\vec{x}+\vec{a})$$

Properties:

$$1 = \langle \hat{T}\psi | \hat{T}\psi \rangle = \langle \psi | \hat{T}^{+}\hat{T} | \psi \rangle \quad \text{hold } \forall\, |\psi\rangle$$

$$\Rightarrow \quad \hat{T}^{+}\hat{T} = 1$$

i) $\hat{T}(\vec{a})$ is a unitary operator.

$$\hat{T}(\vec{a})\,\hat{T}(\vec{b})\,\psi(\vec{x}) = \psi(\vec{x}+\vec{a}+\vec{b})$$

ii) $\hat{T}(\vec{a})\,\hat{T}(\vec{b}) = \hat{T}(\vec{a}+\vec{b})$

Clearly $\quad [\hat{T}(\vec{a}), \hat{T}(\vec{b})] = 0.$

Let's consider an infinitessimal translation : $\vec{a} = \Delta\vec{x}$

$$\hat{T}(\Delta\vec{x})\,\psi(\vec{x}) = \psi(\vec{x}+\Delta\vec{x}) \simeq \psi(\vec{x}) + \nabla\psi(\vec{x})\cdot\Delta\vec{x}$$

$$\hat{T}(\Delta\vec{x})\,\psi(\vec{x}) \cong (\mathbb{1} + \Delta\vec{x}\cdot\nabla)\psi(\vec{x})$$

But $\quad \hat{p} = \frac{\hbar}{i}\nabla$, $\quad$ so

iii) $$\hat{T}(\Delta\vec{x}) \simeq \mathbb{1} + i\,\frac{\Delta\vec{x}\cdot\hat{\vec{p}}}{\hbar}.$$

We can build up a finite translation from lots of infinitessimal translations: $\quad \vec{a} = N\vec{\epsilon}$

$$\hat{T}(\vec{a}) \cong \left(\hat{T}(\vec{\epsilon})\right)^N = \left(\mathbb{1} + \frac{i\,\vec{\epsilon}\cdot\hat{\vec{p}}}{\hbar}\right)^N$$

$$\hat{T}(\vec{a}) = \lim_{N\to\infty}\left(\mathbb{1} + \frac{i\,\vec{a}\cdot\hat{\vec{p}}}{\hbar}\frac{1}{N}\right)^N$$

$$\boxed{\hat{T}(\vec{a}) = \exp\left(i\,\frac{\vec{a}\cdot\hat{\vec{p}}}{\hbar}\right)}$$

$[\hat{T}(\vec{a}), \hat{\vec{p}}] = 0$ $\quad$ but $\quad$ $[\hat{T}(\vec{a}), \hat{\vec{x}}] \neq 0$.

$$[\hat{T}(\vec{a}), \hat{\vec{x}}]\psi(\vec{x}) = (\vec{x}+\vec{a})\psi(\vec{x}+\vec{a}) - \vec{x}\,\psi(\vec{x}+\vec{a})$$

$$= \vec{a}\,\psi(\vec{x}+\vec{a})$$

$$= \vec{a}\,\hat{T}(\vec{a})\psi(\vec{x})$$

$$[\hat{T}(\vec{a}), \hat{\vec{x}}] = \vec{a}\,\hat{T}(\vec{a})$$

Momentum eigenstates are eigenstates of translation:

$$\hat{T}(\vec{a})\,|\vec{p}\rangle = e^{\frac{i\,\vec{a}\cdot\vec{p}}{\hbar}}\,|\vec{p}\rangle$$

Eigenvalue $\lambda = e^{\frac{i\,\vec{a}\cdot\vec{p}}{\hbar}}$.

If the system has a continuous translation symmetry, for example, in the z-direction

$$[\hat{T}(a\hat{z}), \hat{H}] = 0 \quad \forall\, a \in \mathbb{R},$$

then clearly $[\hat{p}_z, \hat{H}] = 0$.

$\hat{p}_z$ is a constant of the motion (conserved quantity).

$$\frac{d}{dt}\langle p_z \rangle = 0.$$

On the other hand, if the system has only a <u>discrete</u> translation symmetry, e.g.,

$$[\hat{T}(a, \hat{x}), \hat{H}] = 0 \quad \text{for a}$$

particular value a_1 (lattice constant),

then $e^{\frac{i a_1 \hat{p}_x}{\hbar}}$ is a constant of the motion, but not $\hat{p}_x$ itself.

$$\frac{d}{dt}\left\langle e^{i\frac{a_1 p_x}{\hbar}} \right\rangle = 0.$$

p_x can only change by $p_x \to p_x + \hbar \frac{2\pi n}{a_1}$, $n \in \mathbb{Z}$

$p_x \bmod \frac{2\pi\hbar}{a_1}$ is conserved.