

## Electron in a uniform static magnetic field

$$H = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2 + qV$$

For static fields, a convenient gauge is the Coulomb gauge

$$\nabla \cdot \vec{A} = 0$$

Recall  $[f(x), p_x] = i\hbar \frac{df}{dx}$

$$\Rightarrow \vec{A} \cdot \vec{p} - \vec{p} \cdot \vec{A} = i\hbar \nabla \cdot \vec{A} = 0$$

The vector potential commutes with momentum in the Coulomb gauge.

$$H = \frac{\vec{p}^2}{2m} - \frac{q}{mc} \vec{A} \cdot \vec{p} + \frac{q^2}{2mc^2} \vec{A}^2 + qV$$

For a constant field  $\vec{B}$ , one can furthermore choose

$$\vec{A} = -\frac{1}{2} \vec{r} \times \vec{B} \quad (\text{symmetric gauge})$$

$\Rightarrow$  Verify  $\nabla \times \vec{A} = \vec{B}$

$$\begin{aligned} \vec{A} \cdot \vec{p} &= -\frac{1}{2} (\vec{r} \times \vec{B}) \cdot \vec{p} = \frac{1}{2} \vec{B} \cdot (\vec{r} \times \vec{p}) \\ &= \frac{1}{2} \vec{B} \cdot \vec{L} \end{aligned} \quad (\text{we used that } [\vec{B}, \vec{r}] = 0)$$

Furthermore,

$$\vec{A}^2 = \frac{1}{4} (\vec{r} \times \vec{B})^2 = \frac{1}{4} [r^2 B^2 - (\vec{r} \cdot \vec{B})^2]$$

$$H = \frac{\vec{p}^2}{2m} - \frac{g}{2mc} \vec{B} \cdot \vec{L}$$

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$$+ \frac{g^2}{8mc^2} [r^2 B^2 - (\vec{r} \cdot \vec{B})^2]$$

$$+ gV$$

For sufficiently small fields, the term linear in  $\vec{B}$  will dominate if  $\langle \vec{r}^2 \rangle$  is bounded. Note that

$$\vec{\mu} = \frac{g}{2mc} \vec{L} \quad = \text{magnetic moment}$$

$$\Delta H = -\vec{\mu} \cdot \vec{B} + \mathcal{O}(B^2)$$

$\Rightarrow$  Normal Zeeman effect

Let's take  $\vec{B} = B \hat{z}$ . (4)

$$H = \frac{\vec{p}^2}{2m} - \frac{gB}{2mc} L_z + \frac{g^2 B^2}{8mc^2} (x^2 + y^2) + gV(\vec{r})$$

The Zeeman splitting is

$$\Delta E = \frac{g\hbar B}{2mc} = \hbar \omega_L$$

$$\omega_L = \frac{eB}{2mc} = \underline{\text{Larmor frequency}}$$

Relative size of linear and quadratic terms

For atomic systems  $\langle x^2 + y^2 \rangle \sim a_0^2$

$$\frac{e^2 B^2 \langle x^2 + y^2 \rangle}{8mc^2} \sim \Delta E_{\text{Zeeman}}$$

$$\frac{e^2 B^2 a_0^2 / 8mc^2}{e\hbar B / 2mc}$$

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$$= \frac{e B a_0^2}{4\hbar c}$$

$$= \frac{e B}{4\hbar c} \left( \frac{\hbar^2}{me^2} \right)^2 = \frac{\hbar^3 B}{4m^2 e^3 c}$$

$$= \left( \frac{\hbar c}{e^2} \right)^2 \frac{\Delta E_{\text{Zeeman}}}{2mc^2} \sim \frac{B}{9 \times 10^9 \text{ gauss}}$$

(Note:  $\frac{e^2}{\hbar c} \sim \frac{1}{137}$ )

fine structure constant

# Landau levels

While the quadratic term in  $\vec{B}$  can be neglected for bounded systems, such as atoms, in weak fields, this is not true for free particles.

Consider a two-dimensional electron gas in the  $x$ - $y$  plane in a field  $\vec{B} = B \hat{z}$ .

Instead of the symmetric gauge, it is convenient, for this problem to work in the Landau gauge

$$\vec{A} = (-yB, 0, 0)$$

$$H = \frac{1}{2m} \left( p_x - \frac{eBy}{c} \right)^2 + \frac{p_y^2}{2m}$$

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$[p_x, H] = 0$ , so seek sol'n of

the form  $\Psi(x, y) = e^{ikx} \chi(y)$ .

$$H \Psi(x, y) = \frac{1}{2m} \left( \hbar k - \frac{eBy}{c} \right)^2 \Psi + \frac{p_y^2}{2m} \Psi$$

$$E \chi(y) = \left[ \frac{p_y^2}{2m} + \frac{1}{2m} \left( \hbar k - \frac{eBy}{c} \right)^2 \right] \chi$$

$$E \chi(y) = \frac{p_y^2}{2m} \chi + \frac{e^2 B^2}{2mc^2} \left( y - \frac{\hbar ck}{eB} \right)^2 \chi$$

$$E \chi = \left[ \frac{p_y^2}{2m} + \frac{m \Omega^2}{2} (y - y_0)^2 \right] \chi$$

$$\Omega = \frac{eB}{mc} = \text{cyclotron frequency}$$

$$y_0 = \frac{\hbar ck}{eB}$$

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$\Rightarrow$  shifted harmonic oscillator

$$E_n = \hbar \Omega \left( n + \frac{1}{2} \right),$$

$$n = 0, 1, 2, \dots, \infty$$

Landau levels

## Degeneracy

suppose system has dimensions

$L_x, L_y$ . Impose periodic

BCs in x-direction.



$$e^{ikL_x} = 1 \Rightarrow k = \frac{2\pi n_x}{L_x}$$

Moreover,  $0 \leq y_0 \leq L_y$

$$0 \leq \frac{\hbar c}{eB} \frac{2\pi n_x}{L_x} \leq L_y$$

$$0 \leq n_x \leq \frac{eB}{\hbar c} L_x L_y$$

Total number of states in a given Landau level of energy  $E_n = \hbar\Omega (n + 1/2)$  is

thus 
$$N_n = \frac{eB}{\hbar c} A$$

$$N_n = \frac{BA}{\phi_0}$$

$$\phi_0 = \frac{hc}{e}$$

(normal flux quantum)

Filling factor  $\nu$

For a total #  $N$  of electrons,  
the total #  $\nu$  of Landau  
levels populated is

$$\nu = \frac{N}{N_n} = \frac{N \phi_0}{BA}$$

$$= \frac{\# \text{ electrons}}{\text{flux quantum}}$$