Physics 570A Lecture 17

The harmonic oscillator revisited: operator method

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Dimensionless variables:

$$\hat{\xi} = \sqrt{\frac{m\omega}{\hbar}}\, \hat{x}, \qquad \hat{\pi} = \frac{1}{i}\frac{d}{d\xi} = \frac{\hat{p}_x}{\sqrt{\hbar m\omega}}$$

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{\pi}^2 + \hat{\xi}^2\right)$$

If $\hat{\pi}$ and $\hat{\xi}$ were numbers, not operators, we could factorize

$$\pi^2 + \xi^2 = (\xi - i\pi)(\xi + i\pi).$$

However, this is not true, because $\hat{\pi}$ and $\hat{\xi}$ do not commute

$$[\hat{\xi}, \hat{\pi}] = \sqrt{\frac{m\omega}{\hbar}} \frac{1}{\sqrt{\hbar m\omega}} [\hat{x}, \hat{p}] = \frac{i\hbar}{\hbar} = i$$

$$(\hat{\xi} - i\hat{\pi})(\hat{\xi} + i\hat{\pi}) = \hat{\pi}^2 + \hat{\xi}^2 + i[\hat{\xi}, \hat{\pi}]$$
$$= \hat{\pi}^2 + \hat{\xi}^2 - \underline{1}$$

Thus

$$\hat{H} = \frac{\hbar\omega}{2}\left((\hat{\xi} - i\hat{\pi})(\hat{\xi} + i\hat{\pi}) + \underline{1}\right)$$

Define

$$\hat{a} = \frac{\hat{\xi} + i\hat{\pi}}{\sqrt{2}} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i\hat{p}_x}{\sqrt{2m\hbar\omega}}$$

$$\hat{a}^{+} = \frac{\hat{\xi} - i\hat{\pi}}{\sqrt{2}} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i\hat{p}_x}{\sqrt{2m\hbar\omega}}$$

Clearly, $\hat{H} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)$.

Let's also define the operator $\hat{N} = \hat{a}^{+}\hat{a}$. If ψ_n is an eigenfunction of $\hat{N}$ with eigenvalue n,

$$\hat{N}\psi_n = n\psi_n,$$

then

$$\hat{H}\psi_n = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n$$

Notice that $\hat{N}$ is hermitian:

$$\hat{N}^{+} = (\hat{a}^{+}\hat{a})^{+} = \hat{a}^{+}(\hat{a}^{+})^{+} = \hat{a}^{+}\hat{a} = \hat{N}.$$

Thus the eigenvalues n must be real. Furthermore $n \geq 0$:

$$n = \langle \psi_n | \hat{N} | \psi_n \rangle = \langle \psi_n | \hat{a}^+ \hat{a} | \psi_n \rangle$$

$$= \langle \phi | \phi \rangle \geq 0,$$

where $\phi = \hat{a} \psi_n$.

To work out the eigenvalues of $\hat{N}$, we first need the commutator

$$[\hat{a}, \hat{a}^+] = \frac{1}{2} [\hat{\xi} + i\hat{\pi}, \hat{\xi} - i\hat{\pi}]$$

$$= \frac{i}{2} \Big(\underbrace{[\hat{\pi}, \hat{\xi}]}_{= -i} - \underbrace{[\hat{\xi}, \hat{\pi}]}_{= i} \Big) = 1$$

$$\boxed{[\hat{a}, \hat{a}^+] = 1}$$

Thus

$$\hat{N}\hat{a} = \hat{a}^{+}\hat{a}\hat{a} = (\hat{a}\hat{a}^{+} - 1)\hat{a} = \hat{a}(\hat{N} - 1)$$

$$\hat{N}\hat{a}\psi_n = \hat{a}(\hat{N} - 1)\psi_n = (n-1)\,\hat{a}\psi_n .$$

Thus $\hat{a}\psi_n$ is an eigenstate of $\hat{N}$ with eigenvalue $n-1$.

If $\langle \psi_n | \psi_n \rangle = 1$, then $\hat{a}\psi_n$ has normalization

$$\langle \hat{a}\psi_n | \hat{a}\psi_n \rangle = \langle \psi_n | \hat{a}^{+}\hat{a} | \psi_n \rangle = n$$

so that

$$\hat{a}\psi_n = \sqrt{n}\,\psi_{n-1} \qquad \text{(up to a phase).}$$

Similarly, $\hat{a}^2\psi_n$ is an eigenfunction of $\hat{N}$ with eigenvalue $n-2$, and

$$\hat{a}^2 \psi_n = \hat{a} \sqrt{n}\, \psi_{n-1} = \sqrt{n(n-1)}\, \psi_{n-2}.$$

Thus if n is an eigenvalue of $\hat{N}$, so are $n-1, n-2, n-3, \ldots$

But this sequence can't keep going indefinitely, because the eigenvalues of $\hat{N}$ are non-negative. There must be a lowest eigenfunction ψ_0,

$$\hat{a}\psi_0 = 0.$$

This is the case if n is an integer! Then using $\hat{a}\psi_n = \sqrt{n}\,\psi_{n-1}$, we get

$$\hat{a}\psi_1 = \sqrt{1}\,\psi_0, \quad \hat{a}\psi_0 = 0.$$

Now let's consider the effect of the operator $\hat{a}^+$.

$$\hat{N}\hat{a}^+ = \hat{a}^+\hat{a}\hat{a}^+ = \hat{a}^+(\hat{a}^+\hat{a} + 1)$$
$$= \hat{a}^+(\hat{N} + 1).$$

$$\hat{N}\hat{a}^+\psi_n = \hat{a}^+(\hat{N}+1)\psi_n = (n+1)\hat{a}^+\psi_n.$$

Thus $\hat{a}^+\psi_n$ is an eigenfunction of $\hat{N}$ with eigenvalue $n+1$.

Thus $n+1$ is also an eigenvalue.

$\hat{N}$ thus has eigenvalues

$0, 1, 2, \ldots \infty$.

Terminology

$\hat{a}$ is known as a <u>lowering operator</u> or <u>annihilation operator</u>, because it lowers the energy of the system by one quantum $\hbar\omega$. $\hat{a}^{+}$ is known as a <u>raising operator</u> or <u>creation operator</u> because it creates an excitation quantum, raising the energy of the system by $\hbar\omega$.

Normalization

$$\langle \hat{a}^\dagger \psi_n | \hat{a}^\dagger \psi_n \rangle = \langle \psi_n | \hat{a} \hat{a}^\dagger | \psi_n \rangle$$

$$= \langle \psi_n | \hat{a}^\dagger \hat{a} + 1 | \psi_n \rangle$$

$$= n + 1$$

$$\Rightarrow \hat{a}^\dagger \psi_n = \sqrt{n+1}\, \psi_{n+1}$$

We can build up all of the eigenstates acting successively with $\hat{a}^\dagger$ on ψ_0:

$$\psi_1 = \hat{a}^\dagger \psi_0$$

$$\psi_2 = \frac{\hat{a}^\dagger}{\sqrt{2}} \psi_1 = \frac{(\hat{a}^\dagger)^2}{\sqrt{2}} \psi_0$$

$$\psi_3 = \frac{\hat{a}^\dagger}{\sqrt{3}} \psi_2 = \frac{(\hat{a}^\dagger)^3}{\sqrt{3!}} \psi_0$$

$$\vdots$$

In general,

$$\psi_n = \frac{(\hat{a}^+)^n}{\sqrt{n!}}\,\psi_0 .$$

It should be emphasized that all of this follows from the operator algebra $[\hat{a}, \hat{a}^+] = 1$.

Explicitly,

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{\xi} + i\hat{\pi}) = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)$$

$$\hat{a}\,\psi_0 = 0$$

$$\left(\xi + \frac{d}{d\xi}\right)\psi_0(\xi) = 0$$

$$\frac{d\psi_0}{d\xi} = -\xi\,\psi_0$$

$$\frac{d\psi_0}{\psi_0} = -\xi \, d\xi$$

$$\ln \psi_0 = -\frac{\xi^2}{2} + C, \qquad C = \ln A_0$$

$$\psi_0(\xi) = A_0 e^{-\xi^2/2}, \qquad A_0 = \left(\frac{1}{\sqrt{\pi}}\right)^{1/2}.$$

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n!}} \left(\xi - \frac{d}{d\xi}\right)^n \psi_0(\xi)$$

$$= \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}.$$

These are the same wavefunctions we obtained by solving Schrödinger's equation. This leads to an alternative formula for the Hermite polynomials

$$H_n(\xi) = e^{\xi^2/2} \left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}.$$