Physics 570A Lecture 19

WKB Approximation II

Condition for validity of WKB:

$$\frac{A''(x)}{A(x)} \ll k(x)^2 = \frac{2m}{\hbar^2}\left(E - V(x)\right)$$

$$A(x) = \frac{C}{\sqrt{k(x)}}$$

$$A'(x) = -\frac{1}{2} C k^{-3/2} k'(x)$$

$$A''(x) = \frac{3}{4} C k^{-5/2} (k'(x))^2 - \frac{1}{2} C k^{-3/2} k''(x)$$

$$\frac{A''}{A} = \frac{3}{4}\left(\frac{k'}{k}\right)^2 - \frac{1}{2}\frac{k''(x)}{k}$$

But $\quad 2kk' = -\frac{2m}{\hbar^2} V'(x)$

$$(k')^2 + kk'' = -\frac{m}{\hbar^2} V''(x).$$

Thus, WKB is valid provided the potential varies slowly:

$$\frac{\lambda}{E}\left|\frac{\partial V}{\partial x}\right| \ll 1, \quad \frac{\lambda^2}{E}\left|\frac{\partial^2 V}{\partial x^2}\right| \ll 1.$$

Turning points

Note that the WKB wavefunction is undefined at a classical turning point, where $k(x) = 0$. WKB predicts that

$$\rho(x) = \frac{|C|^2}{|k(x)|} \to \infty$$ at a turning point.

[Does this make sense, classically?]

However, the potential can be linearized at a turning point, and an exact soln of the Sch. eq. can be obtained:

$$V(x) \cong E + \underbrace{\left.\frac{\partial V}{\partial x}\right|_{x_0}}_{\alpha} (x - x_0)$$

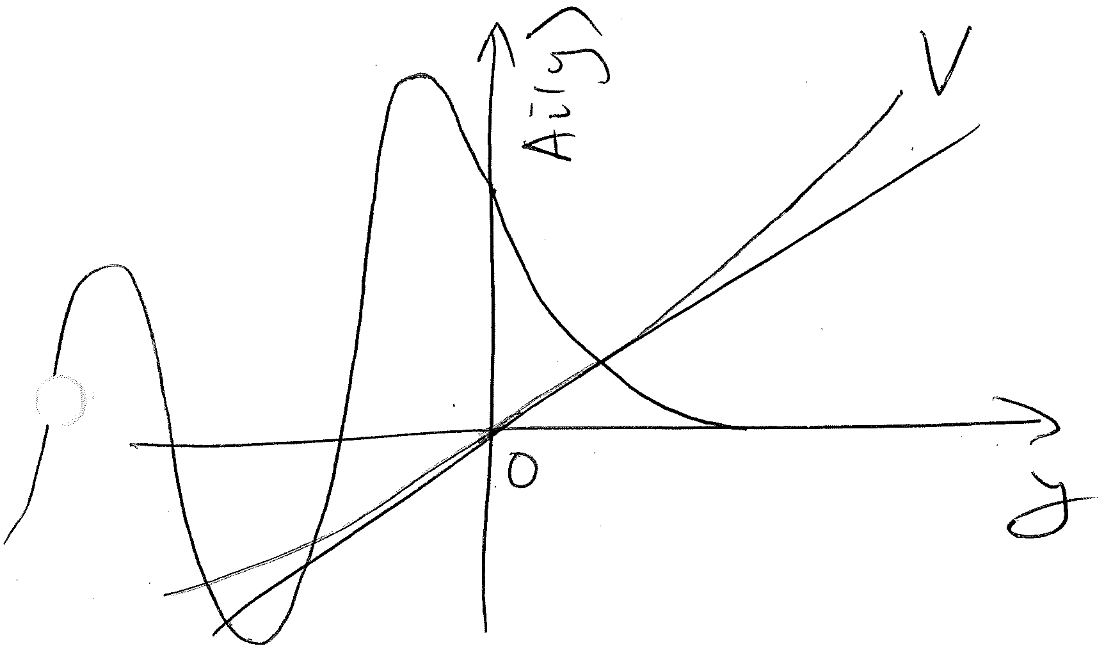
$$\Rightarrow \frac{d^2\psi}{dx^2} - \frac{2m\alpha}{\hbar^2}(x - x_0)\psi = 0$$

Let $y = \left(\frac{2m\alpha}{\hbar^2}\right)^{1/3}(x - x_0)$

$$\Rightarrow \frac{d^2\psi}{dy^2} - y\psi = 0$$

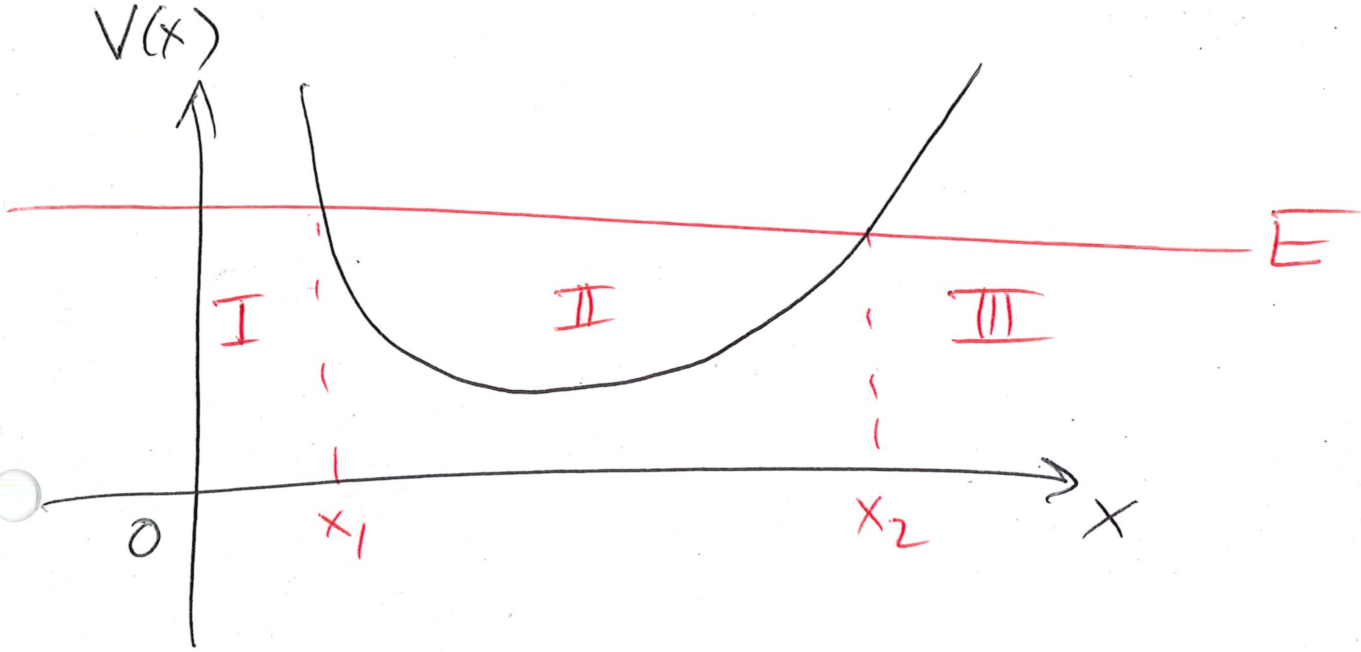
The solution of this eq. is the Airy function $Ai(y)$

<u>Connection formulas (for bound states)</u>



For bound states, we are interested in the Airy function that decays to zero as $y \to \infty$, and is oscillatory for $y < 0$, whose asymptotic behavior is

$$Ai(y) \sim \begin{cases} \dfrac{1}{2\sqrt{\pi}\, y^{1/4}} e^{-\frac{2}{3} y^{3/2}}, & y > 0 \\[2ex] \dfrac{1}{\sqrt{\pi}\,(-y)^{1/4}} \sin\left(\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4}\right), & y < 0 \end{cases}$$



$$\psi_{I}(x) = \frac{C_{I}}{\sqrt{|k(x)|}} e^{\int_{x_1}^{x} |k(x')|\, dx'}$$

$$\psi_{III}(x) = \frac{C_{III}}{\sqrt{|k(x)|}} e^{-\int_{x_2}^{x} |k(x')|\, dx'}$$

In the neighborhood of the turning point $x_0 = x_2$, we have

$$p^2 = 2m(E-V) = -2m\alpha(x-x_2)$$

$$= -(2m\alpha\hbar)^{2/3} y$$

$$\Rightarrow \; -\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|\,dx' = -\int_0^y \sqrt{y'}\,dy'$$

$$= -\frac{2}{3} y^{3/2}$$

Thus, the WKB wavefunction agrees with the asymptotic form of the Airy function to the right of x_2. On the other hand, the integrand in the exponent of the oscillatory functions in region II is

$$\frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' = -\int_0^y \sqrt{-y'}\,dy' = \frac{2}{3}(-y)^{3/2}$$

In order to match up at $x = x_2$,

we must take

$$\psi_{II}(x) = \frac{2C_{III}}{\sqrt{|k(x)|}} \sin\left(\int_x^{x_2} k(x')\,dx' + \frac{\pi}{4}\right).$$

Similarly, matching solutions at

$x = x_1$ gives

$$\psi_{II}(x) = \frac{2C_{I}}{\sqrt{|k(x)|}} \sin\left(\int_{x_1}^{x} k(x')\,dx' + \frac{\pi}{4}\right)$$

The two expressions for ψ_{II} must be equal:

$$C_I \sin\left(\int_{x_1}^{x} k(x')\,dx' + \frac{\pi}{4}\right) = C_{III} \sin\left(\int_{x}^{x_2} k(x')\,dx' + \frac{\pi}{4}\right)$$

But $\int_{x_1}^{x} k(x')\,dx' = \int_{x_1}^{x_2} k(x')\,dx' - \int_{x}^{x_2} k(x')\,dx'$.

we require

$$\int_{x_1}^{x_2} k(x')\,dx' = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, \ldots$$

The condition that a bound state must satisfy is thus

$$\int_{x_1}^{x_2} k(x')\,dx' = \frac{1}{\hbar}\int_{x_1}^{x_2} dx' \sqrt{2m(E - V(x'))} = \left(n + \frac{1}{2}\right)\pi$$

or $$\oint k(x')\,dx' = \left(n+\tfrac{1}{2}\right)2\pi$$

$$\oint p\,dx = \left(n+\tfrac{1}{2}\right)h$$

This is the semiclassical (Bohr-Sommerfeld) quantization condition.