# Lecture 22 Feynman's path integral approach

Dirac: $\langle x', t' | x, t \rangle \sim e^{i \frac{S_{cl}(x', t'; x, t)}{\hbar}}$

Feynman: $\langle x', t' | x, t \rangle = \sum_{\text{paths}} e^{i \frac{S_{cl}[x(t)]}{\hbar}}$

$\langle x', t' | x, t \rangle$ is the <u>propagator</u>, the amplitude to propagate from the point $x$ at time $t$ to the point $x'$ at time $t'$. Here, the bras and kets are position eigenstates in the Heisenberg representation.

$|x, t\rangle = e^{i \frac{\hat{H} t}{\hbar}} |x\rangle$, where $|x\rangle$ is the position eigenket in the Schrödinger rep.

$$\hat{x}(t) |x, t\rangle = x |x, t\rangle$$

$$e^{i\frac{\hat{H}t}{\hbar}} \hat{x} \left( e^{-i\frac{\hat{H}t}{\hbar}} |x, t\rangle \right) = x |x, t\rangle$$

$$\hat{x}\underbrace{\left(e^{\frac{-i\hat{H}t}{\hbar}}|x,t\rangle\right)}_{|x\rangle} = x\left(e^{\frac{-i\hat{H}t}{\hbar}}|x,t\rangle\right)$$

$$\langle x',t'|x,t\rangle = \langle x'|e^{\frac{-i\hat{H}t'}{\hbar}}e^{\frac{i\hat{H}t}{\hbar}}|x\rangle$$

$$= \langle x'|e^{\frac{-i\hat{H}(t'-t)}{\hbar}}|x\rangle.$$

The propagator is just a matrix element of the time-evolution operator in the position basis. Although this looks quite straightforward, it is in general non-trivial to evaluate since typically

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad \text{and} \quad [\hat{x},\hat{p}] \neq 0$$

and $[\hat{p}^2, V(\hat{x})] \neq$ const. so we can't use Weyl's identity. Following the book, let us denote

$$K(x',t';x,t) \equiv \langle x',t'|x,t\rangle.$$

Important properties

$$|\psi(t')\rangle = \hat{U}(t'-t)|\psi(t)\rangle$$

$$\langle x'|\psi(t')\rangle = \int dx\, \langle x'|\hat{U}(t'-t)|x\rangle\langle x|\psi(t)\rangle$$

$$\boxed{\psi(x',t') = \int dx\, K(x',t';x,t)\,\psi(x,t)}$$

$K$ is closely related to $G^r$ : $(t' \geq t)$

$$K(x',t';x,t) = i\hbar\, G^r(x',t';x,t)$$

$$e^{-i\frac{\hat{H}(t'-t)}{\hbar}} = e^{-i\frac{\hat{H}(t'-t_1)}{\hbar}}\, e^{-i\frac{\hat{H}(t_1-t)}{\hbar}}$$

$$\langle x'|e^{-i\frac{\hat{H}(t'-t)}{\hbar}}|x\rangle = \int dx_1\, \langle x'|e^{-i\frac{\hat{H}(t'-t_1)}{\hbar}}|x_1\rangle \times \langle x_1|e^{-i\frac{\hat{H}(t_1-t)}{\hbar}}|x\rangle$$

$$\boxed{K(x',t';x,t) = \int dx_1\, K(x',t';x_1,t_1)\,K(x_1,t_1;x,t)}$$

Let's use the later property to ~~brea~~ break up the time interval $t'-t$ into many infinitessimal pieces:

$$N\Delta t = t' - t \qquad t_n = t + n\Delta t$$

$$t_0 = t$$

$$t_N = t'$$

$$K(x',t';x,t) = \int dx_{N-1}\, dx_{N-2} \cdots dx_1\, K(x',t';x_{N-1},t_{N-1})$$

$$\times K(x_{N-1},t_{N-1};x_{N-2},t_{N-2}) \times \cdots K(x_1,t_1;x,t)$$

$$K(x_n,t_n;x_{n-1},t_{n-1}) = \langle x_n | e^{-i\frac{\hat{H}\Delta t}{\hbar}} | x_{n-1} \rangle$$

Suppose $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$.

$$e^{-i\frac{\hat{H}\Delta t}{\hbar}} = e^{-i\frac{\hat{p}^2\Delta t}{2m\hbar}} e^{-i\frac{V(\hat{x})\Delta t}{\hbar}} + \mathcal{O}(\Delta t)^2$$

$$\langle x_n | e^{-i\frac{\hat{H}\Delta t}{\hbar}} | x_{n-1} \rangle = \int \frac{dp_n}{h} \langle x_n | p_n \rangle \langle p_n | e^{-i\frac{\hat{H}\Delta t}{\hbar}} | x_{n-1} \rangle$$

$$= \int \frac{dp_n}{h} e^{\frac{i p_n x_n}{\hbar}} e^{-\frac{i p_n^2 \Delta t}{2m\hbar}} e^{-i\frac{V(x_{n-1})\Delta t}{\hbar}} e^{-i\frac{p_n x_{n-1}}{\hbar}} + \mathcal{O}(\Delta t)^2$$

$$= \int \frac{dp_n}{h} e^{\frac{i p_n (x_n - x_{n-1})}{\hbar} - \frac{i}{\hbar}\left[\frac{p_n^2}{2m} + V(x_{n-1})\right]\Delta t} + \mathcal{O}(\Delta t)^2$$

$$= \int \frac{dp_n}{h} e^{\frac{i}{\hbar}\left[p_n \frac{x_n - x_{n-1}}{\Delta t} - \frac{p_n^2}{2m} - V(x_{n-1})\right]\Delta t} + \mathcal{O}(\Delta t)^2$$

Thus,

$$K(x',t';x,t) = \int \frac{dp_N}{h} \prod_{n=1}^{N-1} \frac{dp_n\, dx_n}{h} e^{\frac{i}{\hbar}\left[p_{n+1}\frac{x_{n+1}-x_n}{\Delta t} - \left\{\frac{p_{n+1}^2}{2m} + V(x_n)\right\}\right]\Delta t} + \mathcal{O}(\Delta t)^2$$

$$\boxed{\begin{aligned} K(x',t';x,t) &= \lim_{N\to\infty} \frac{1}{h^N} \int dp_N \prod_{n=1}^{N-1} dp_n\, dx_n\, e^{\frac{i}{\hbar}\left[p_n \dot{x}_n - H(x_n,p_n)\right]\Delta t} \\ &= \int \mathcal{D}[p(t)]\, \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}\int_t^{t'} dt'' \left[p(t'')\dot{x}(t'') - H(x(t''),p(t''))\right]} \end{aligned}}$$

This is the phase-space version of the Feynman path integral. The p-integrals can be performed explicitly for the case at hand:

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{i\hbar \Delta t}}\; e^{\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x_n - x_{n-1})^2}{\Delta t} - V(x_{n-1})\Delta t\right)} + \mathcal{O}(\Delta t)^2$$

$$= \sqrt{\frac{m}{i\hbar \Delta t}}\; e^{\frac{i}{\hbar} S_{cl}(x_n, t_n; x_{n-1}, t_{n-1})} + \mathcal{O}(\Delta t)^2$$

Here $S_{cl} = \int_{t_{n-1}}^{t_n} dt\, L\left(x(t), \frac{dx}{dt}\right)$,

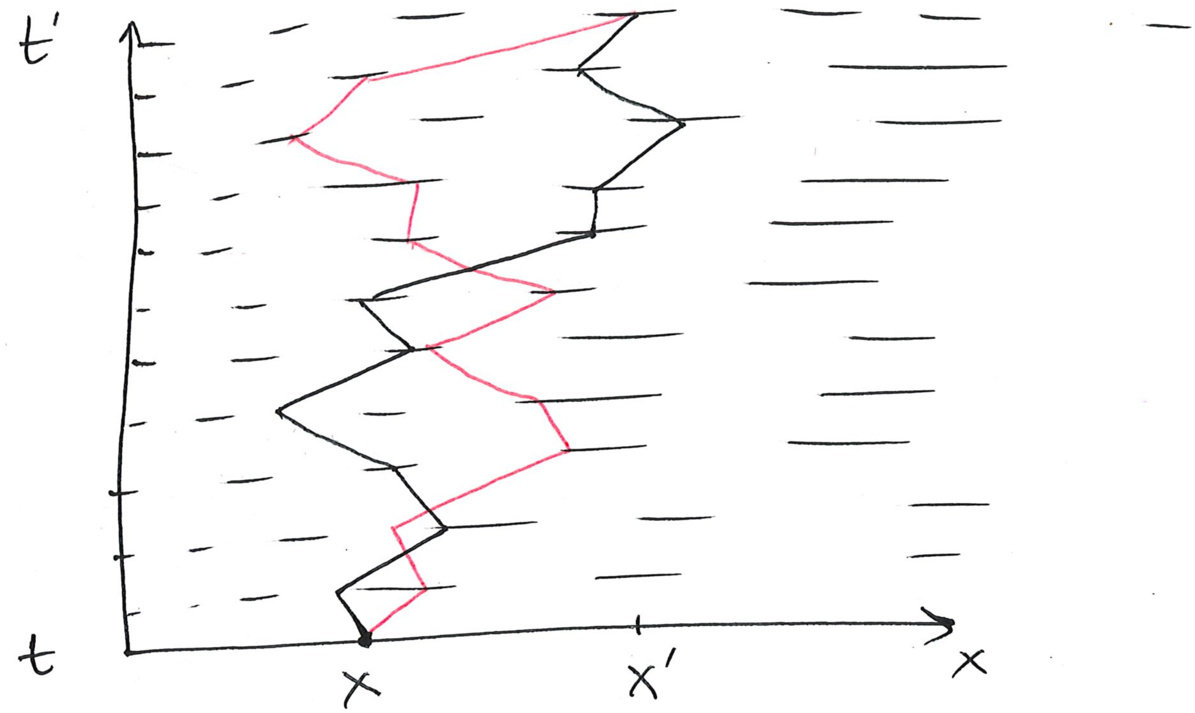
with $x_n = x(t_n)$ and $x_{n-1} = x(t_{n-1})$,

and $L = \frac{m\dot{x}^2}{2} - V(x)$ is the classical Lagrangian. Thus

$$\boxed{K(x', t'; x, t) = \int_{x = x(t)}^{x' = x(t')} \mathcal{D}[x(t)]\; e^{\frac{i}{\hbar}\int_t^{t'} dt''\, L(x(t''), \dot{x}(t''))}}$$

Here $\mathcal{D}[x(t)] = \lim_{N\to\infty} \prod_{n=1}^{N-1} \sqrt{\frac{mN}{i\hbar(t'-t)}} \int_{-\infty}^{\infty} dx_n$.

The sum is over "Brownian" trajectories with piecewise constant velocity:




<u>Example: free particle</u>

$$K(x',t';x,t) = \sqrt{\frac{m}{i\hbar(t'-t)}}\, e^{\frac{im(x'-x)^2}{2\hbar(t'-t)}}$$

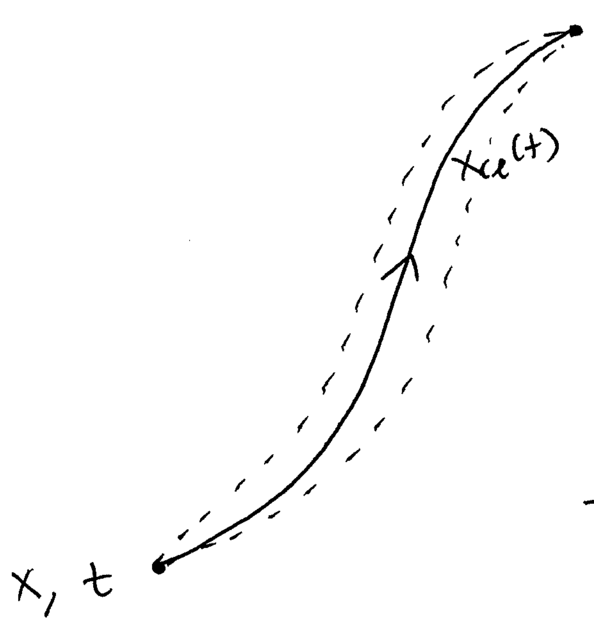
In general, the path integral is difficult to evaluate in closed form. However, it is useful for computation, especially in its Euclidean (imaginary time) form.

Also, it is very useful for deriving the semiclassical approximation.

$$K = \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]}.$$

In the semiclassical limit $\frac{S}{\hbar} \gg 1$, the integrand oscillates rapidly, and the integral may be approximated by the <u>method of stationary phase</u>.

It will be dominated by the path(s) for which $\delta S[x(t)] = 0$, where the action is an extremum, and a thin manifold of paths in its vicinity. This is just the <u>classical trajectory</u>.




paths important in the limit $\hbar \to 0$.

cf. <u>stationary phase approximation</u>

$$I(\alpha) = \int_{-\infty}^{\infty} dx \; e^{i \alpha \theta(x)} .$$

What happens when $\alpha \to \infty$? Integrand oscillates rapidly, averaging to zero everywhere, except where the phase is stationary: $\theta'(x_0) = 0$

$$\theta(x) \simeq \theta(x_0) + \frac{1}{2} \theta''(x_0)(x - x_0)^2 + \cdots$$

$$I(\alpha) \underset{\alpha \to \infty}{\sim} e^{i \alpha \theta(x_0)} \int_{-\infty}^{\infty} dx \; e^{i \frac{\alpha}{2} \theta''(x_0)(x - x_0)^2}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}$$

Analytic continuation:

$$\int_{-\infty}^{\infty} dx \quad e^{i \frac{\alpha}{2} \theta''(x_0)(x-x_0)^2} = \sqrt{\frac{-2\pi}{i\alpha\theta''(x_0)}}$$

$$I(\alpha) \underset{\alpha\to\infty}{\sim} \sqrt{\frac{2\pi}{i\alpha\theta''(x_0)}} \; e^{i\alpha\theta(x_0) + i\frac{\pi}{2}}$$

This is analogous to the semiclassical limit of the Feynman path integral, where $\hbar = \frac{1}{\alpha} \to 0$ and $S_{cl} = \theta(x_0)$.