

# Lecture 22 Feynman's path integral approach

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$$\text{Dirac: } \langle x', t' | x, t \rangle \sim e^{i \frac{S_{cl}(x', t'; x, t)}{\hbar}}$$

$$\text{Feynman: } \langle x', t' | x, t \rangle = \sum_{\text{paths}} e^{i \frac{S_{cl}[x(t)]}{\hbar}}$$

$\langle x', t' | x, t \rangle$  is the propagator, the amplitude to propagate from the point  $x$  at time  $t$  to the point  $x'$  at time  $t'$ . Here, the bras and kets are position eigenstates in the Heisenberg representation.

$|x, t\rangle = e^{i \frac{\hat{H}t}{\hbar}} |x\rangle$ , where  $|x\rangle$  is the position eigenket in the Schrödinger rep.

$$\hat{X}(t) |x, t\rangle = x |x, t\rangle$$

$$e^{i \frac{\hat{H}t}{\hbar}} \hat{X} \left( e^{-i \frac{\hat{H}t}{\hbar}} |x, t\rangle \right) = x |x, t\rangle$$

$$\hat{x} \left( \underbrace{e^{-i\frac{\hat{H}t}{\hbar}} |x, t\rangle}_{|x\rangle} \right) = x \left( e^{-i\frac{\hat{H}t}{\hbar}} |x, t\rangle \right)$$

$$\begin{aligned} \langle x', t' | x, t \rangle &= \langle x' | e^{-i\frac{\hat{H}t'}{\hbar}} e^{i\frac{\hat{H}t}{\hbar}} |x\rangle \\ &= \langle x' | e^{-i\frac{\hat{H}(t'-t)}{\hbar}} |x\rangle. \end{aligned}$$

The propagator is just a matrix element of the time-evolution operator in the position basis. Although this looks quite straightforward, it is in general non-trivial to evaluate since typically

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad \text{and} \quad [\hat{x}, \hat{p}] \neq 0$$

and  $[\hat{p}^2, V(\hat{x})] \neq \text{const.}$  so we can't use Weyl's identity. Following the book, let

us denote

$$K(x', t'; x, t) \equiv \langle x', t' | x, t \rangle.$$

## Important properties

$$|\psi(t')\rangle = \hat{U}(t'-t) |\psi(t)\rangle$$

$$\langle x' | \psi(t') \rangle = \int dx \langle x' | \hat{U}(t'-t) | x \rangle \langle x | \psi(t) \rangle$$

$$\boxed{\psi(x', t') = \int dx K(x', t'; x, t) \psi(x, t)}$$

$K$  is closely related to  $G^r$ : ( $t' \geq t$ )

$$K(x', t'; x, t) = \frac{1}{i\hbar} G^r(x', t'; x, t)$$

$$e^{-i \frac{\hat{H}(t'-t)}{\hbar}} = e^{-i \frac{\hat{H}(t'-t_1)}{\hbar}} e^{-i \frac{\hat{H}(t_1-t)}{\hbar}}$$

$$\langle x' | e^{-i \frac{\hat{H}(t'-t)}{\hbar}} | x \rangle = \int dx_1 \langle x' | e^{-i \frac{\hat{H}(t'-t_1)}{\hbar}} | x_1 \rangle \times \langle x_1 | e^{-i \frac{\hat{H}(t_1-t)}{\hbar}} | x \rangle$$

$$\boxed{K(x', t'; x, t) = \int dx_1 K(x', t'; x_1, t_1) K(x_1, t_1; x, t)}$$

Let's use the later property to break up the time interval  $t' - t$  into many infinitesimal pieces:

$$N \Delta t = t' - t \quad t_n = t + n \Delta t$$

$$t_0 = t$$

$$t_N = t'$$

$$K(x', t'; x, t) = \int dx_{N-1} dx_{N-2} \dots dx_1 K(x', t'; x_{N-1}, t_{N-1})$$

$$\times K(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \times \dots \times K(x_1, t_1; x, t)$$

$$K(x_n, t_n; x_{n-1}, t_{n-1}) = \langle x_n | e^{-i \frac{\hat{H} \Delta t}{\hbar}} | x_{n-1} \rangle$$

Suppose  $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ .

$$e^{-i \frac{\hat{H} \Delta t}{\hbar}} = e^{-i \frac{\hat{p}^2 \Delta t}{2m \hbar}} e^{-i \frac{V(\hat{x}) \Delta t}{\hbar}} + \mathcal{O}(\Delta t)^2$$

$$\langle x_n | e^{-i \frac{\hat{H} \Delta t}{\hbar}} | x_{n-1} \rangle = \int \frac{dp_n}{\hbar} \langle x_n | p_n \rangle \langle p_n | e^{-i \frac{\hat{H} \Delta t}{\hbar}} | x_{n-1} \rangle$$

$$= \int \frac{dp_n}{h} e^{i \frac{p_n x_n}{h}} e^{-i \frac{p_n^2 \Delta t}{2m h}} e^{-i \frac{V(x_{n-1}) \Delta t}{h}} e^{-i \frac{p_n x_{n-1}}{h}} + \mathcal{O}(\Delta t)^2$$

$$= \int \frac{dp_n}{h} e^{i \frac{p_n (x_n - x_{n-1})}{h} - \frac{i}{h} \left[ \frac{p_n^2}{2m} + V(x_{n-1}) \right] \Delta t} + \mathcal{O}(\Delta t)^2$$

$$= \int \frac{dp_n}{h} e^{\frac{i}{h} \left[ p_n \frac{x_n - x_{n-1}}{\Delta t} - \frac{p_n^2}{2m} - V(x_{n-1}) \right] \Delta t} + \mathcal{O}(\Delta t)^2$$

Thus,

$$K(x', t'; x, t) = \int \frac{dp_N}{h} \prod_{n=1}^{N-1} \frac{dp_n dx_n}{h} e^{\frac{i}{h} \left[ p_{n+1} \frac{x_{n+1} - x_n}{\Delta t} - \left\{ \frac{p_{n+1}^2}{2m} + V(x_n) \right\} \Delta t \right]} + \mathcal{O}(\Delta t)^2$$

$$K(x', t'; x, t) = \lim_{N \rightarrow \infty} \frac{1}{h^N} \int dp_N \prod_{n=1}^{N-1} \int dp_n dx_n e^{\frac{i}{h} [p_n \dot{x}_n - H(x_n, p_n)] \Delta t}$$

$$\equiv \int \mathcal{D}[p(t)] \mathcal{D}[x(t)] e^{\frac{i}{h} \int_t^{t'} dt'' [p(t'') \dot{x}(t'') - H(x(t''), p(t''))]}$$

This is the phase-space version of the Feynman path integral. The  $p$ -integrals can be performed explicitly for the case at hand:

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{i\hbar\Delta t}} e^{\frac{i}{\hbar} \left( \frac{m}{2} \frac{(x_n - x_{n-1})^2}{\Delta t} - V(x_{n-1}) \Delta t \right)} + \mathcal{O}(\Delta t)^2$$

$$= \sqrt{\frac{m}{i\hbar\Delta t}} e^{\frac{i}{\hbar} S_{cl}(x_n, t_n; x_{n-1}, t_{n-1})} + \mathcal{O}(\Delta t)^2$$

Here  $S_{cl} = \int_{t_{n-1}}^{t_n} dt L(x(t), \frac{dx}{dt})$ ,

with  $x_n = x(t_n)$  and  $x_{n-1} = x(t_{n-1})$ ,

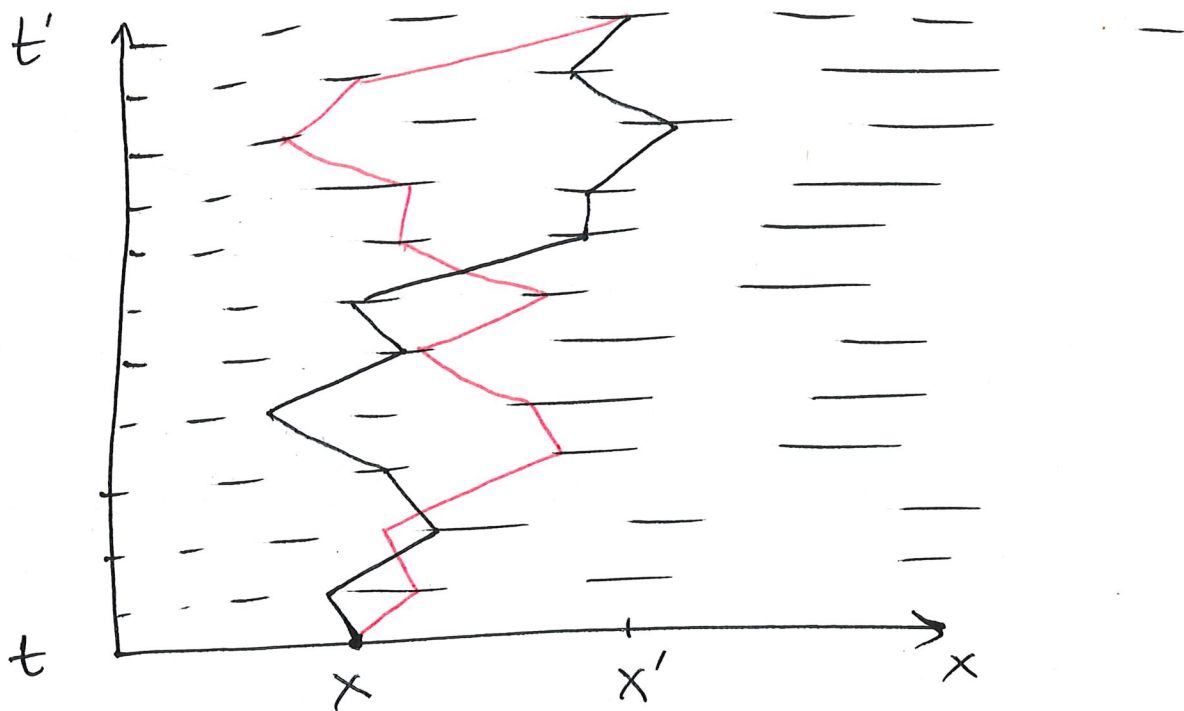
and  $L = \frac{m\dot{x}^2}{2} - V(x)$  is the

classical Lagrangian. Thus

$$K(x', t'; x, t) = \int_{x=x(t)}^{x'=x(t')} \mathcal{D}[x(t)] e^{\frac{i}{\hbar} \int_t^{t'} dt'' L(x(t''), \dot{x}(t''))}$$

Here  $\int \mathcal{D}[x(t)] = \lim_{N \rightarrow \infty} \prod_{n=1}^{N-1} \sqrt{\frac{mN}{i\hbar(t'-t)}} \int_{-\infty}^{\infty} dx_n$ .

The sum is over "Brownian" trajectories with piecewise constant velocity:



Example: free particle

$$K(x', t'; x, t) = \sqrt{\frac{m}{i\hbar(t'-t)}} e^{iM \frac{(x'-x)^2}{2\hbar(t'-t)}}$$

In general, the path integral is difficult to evaluate in closed form. However,

$i\hbar$  is useful for computation, especially in its Euclidean (imaginary time) form.

Also,  $i\hbar$  is very useful for deriving the semiclassical approximation.

$$K = \int \mathcal{D}[x(t)] e^{\frac{i}{\hbar} S[x(t)]}.$$

In the semiclassical limit  $\frac{S}{\hbar} \gg 1$ ,

the integrand oscillates rapidly, and

the integral may be approximated

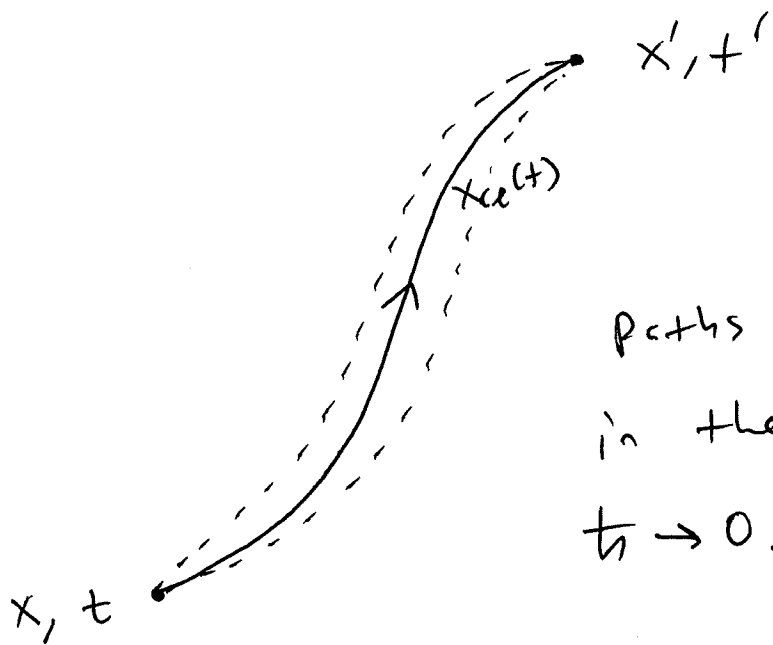
by the method of stationary phase.

It will be dominated by the path(s)

for which  $\delta S[x(t)] = 0$ , where the action is an extremum, and a thin manifold of paths in its vicinity. This is

just the classical trajectory.





paths important  
in the limit  
 $\hbar \rightarrow 0$ .

cf. stationary phase approximation

$$I(\alpha) = \int_{-\infty}^{\infty} dx e^{i\alpha\theta(x)}$$

What happens when  $\alpha \rightarrow \infty$ ? Integrand oscillates rapidly, averaging to zero everywhere, except where the phase is stationary:  $\theta'(x_0) = 0$ .

$$\theta(x) \approx \theta(x_0) + \frac{1}{2}\theta''(x_0)(x-x_0)^2 + \dots$$

$$I(\alpha) \underset{\alpha \rightarrow \infty}{\sim} e^{i\alpha\theta(x_0)} \int_{-\infty}^{\infty} dx e^{i\frac{\alpha}{2}\theta''(x_0)(x-x_0)^2}$$

$$\int_{-\infty}^{\infty} dx e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}$$

Analytic continuation:

$$\int_{-\infty}^{\infty} dx e^{i \frac{\alpha}{2} \theta''(x_0) (x-x_0)^2} = \sqrt{\frac{-2\pi}{i\alpha \theta''(x_0)}}$$

$$I(\alpha) \underset{\alpha \rightarrow \infty}{\sim} \sqrt{\frac{2\pi}{i\alpha \theta''(x_0)}} e^{i\alpha \theta(x_0) + i\frac{\pi}{2}}$$

This is analogous to the semiclassical limit of the Feynman path integral, where  $\hbar = \frac{1}{\alpha} \rightarrow 0$  and  $S_{cl} = \theta(x_0)$ .