

The angular momentum matrices

$$\vec{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$$

$$L_z Y_{lm} = \hbar m Y_{lm}$$

$$\Rightarrow \langle l'm' | \vec{L}^2 | lm \rangle = \hbar^2 l(l+1) \delta_{ll'} \delta_{mm'}$$

$$\langle l'm' | L_z | lm \rangle = \hbar m \delta_{ll'} \delta_{mm'}$$

form a complete, orthonormal basis in that subspace. 4

The operators L_{\pm} , L_x , and L_y can also be represented by matrices:

$$L_{\pm} Y_{\ell m} = \hbar \sqrt{\ell(\ell+1) - m(m\pm 1)} Y_{\ell, m\pm 1}$$

$$\langle \ell' m' | L_+ | \ell m \rangle = \hbar \sqrt{\ell(\ell+1) - m(m+1)} \delta_{\ell \ell'} \delta_{m', m+1}$$

$$\langle \ell' m' | L_- | \ell m \rangle = \hbar \sqrt{\ell(\ell+1) - m(m-1)} \delta_{\ell \ell'} \delta_{m', m-1}$$

In the subspace with $\ell=1$, L_{\pm} are 3×3 matrices

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

In this matrix representation, 5
the eigenfunctions Y_{lm} become
column vectors with $2l+1$ components

$$Y_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_{10} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Y_{1,-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L_+ Y_{11} = 0 \quad L_+ Y_{10} = \hbar\sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\sqrt{2} Y_{11}$$

$$L_+ Y_{1,-1} = \hbar\sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{2} Y_{10}, \text{ etc.}$$

In the subspace with $l=1$,
the matrices for L_x and
 L_y are

$$(L_x) = \frac{1}{2} (L_+) + \frac{1}{2} (L_-) = \hbar \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$(L_y) = \frac{1}{2i} (L_+) - \frac{1}{2i} (L_-) = \hbar \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Spin - 1/2

It is also possible to construct matrix representations for half-odd integer values of the

angular momentum quantum #, although there are no spherical harmonics in that case.

The case where the total angular momentum quantum # is $1/2$ is of particular interest, since it represents the intrinsic angular momentum, or spin, of many elementary particles, such as electrons, quarks, etc. (also protons and neutrons). For spin, the angular momentum quantum # is denoted by s :

$$S^2 \psi_{sm} = \hbar^2 s(s+1) \psi_{sm}$$

$$S_z \psi_{sm} = \hbar m \psi_{sm}$$

where ψ_{sm} is a spin wavefunction to be determined.

$$\underline{S = 1/2}$$

$$S(S+1) = \frac{1}{2} \left(\frac{3}{2} \right) = \frac{3}{4} \quad \left. \vphantom{\frac{1}{2}} \right\} 8$$

$$m = \pm 1/2$$

$$\text{Let } \Psi_{\frac{1}{2}, \frac{1}{2}} = \Psi_{\uparrow}, \quad \Psi_{\frac{1}{2}, -\frac{1}{2}} = \Psi_{\downarrow}.$$

$$S_z \Psi_{\uparrow} = \frac{\hbar}{2} \Psi_{\uparrow}$$

$$S_z \Psi_{\downarrow} = -\frac{\hbar}{2} \Psi_{\downarrow}$$

$$S_+ \Psi_{\downarrow} = \hbar \sqrt{S(S+1) - m(m+1)} \Psi_{\uparrow}$$

$$= \hbar \sqrt{\frac{3}{4} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2} + 1\right)} \Psi_{\uparrow}$$

$$= \hbar \Psi_{\uparrow}$$

Similarly,

$$S_- \Psi_{\uparrow} = \hbar \Psi_{\downarrow}$$

Matrix representations

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$$(\vec{S})^2 = \hbar^2 \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}$$

$$(S_z) = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$(S_+) = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(S_-) = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\psi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(S_x) = \frac{1}{2} (S_+) + \frac{1}{2} (S_-)$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(S_y) = \frac{1}{2i} (S_+) - \frac{1}{2i} (S_-)$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The Pauli matrices are

defined by $S = \frac{\hbar}{2} \sigma$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties:

$$\sigma_i^2 = \mathbb{1}$$

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$$[\sigma_x, \sigma_y] = 2i\sigma_z \quad (\text{and cyclic permutations})$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbb{1}$$

(anticommutation)

Eigenvectors of σ_x

$$\sigma_x \psi = \lambda \psi$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$$

$$\Rightarrow \lambda = \pm 1 \quad (\text{eigenvalues})$$

$$\begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix} \begin{pmatrix} a_{\pm} \\ b_{\pm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\mp a_{\pm} + b_{\pm} = 0$$

$$b_{\pm} = \pm a_{\pm}$$

$$\Psi_{S_x = \pm \hbar/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\Psi_{\uparrow} \pm \Psi_{\downarrow})$$