Magnetic moment

Definition (classical)

$$\vec{\mu} = \frac{1}{2c}\int d^3r \left(\vec{r} \times \vec{J}_q(\vec{r}) \right)$$

QM: $\vec{J}_q = \frac{q}{2M}\left(\psi^* \frac{\hbar}{i}\nabla\psi - \psi \frac{\hbar}{i}\nabla\psi^* \right)$

$$= \frac{q}{M} \mathrm{Re}\left[\psi^* \vec{p}\, \psi\right]$$

$$\langle \vec{\mu} \rangle = \frac{q}{2Mc}\int d^3r \left(\vec{r} \times \mathrm{Re}\left[\psi^* \vec{p}\, \psi\right] \right)$$

$$= \frac{q}{2Mc} \mathrm{Re} \int d^3r \; \psi^* (\vec{r} \times \vec{p})\, \psi$$

$$= \frac{q}{2Mc} \langle \vec{L} \rangle$$

Magnetic moment operator

$$\vec{\mu} = \frac{q}{2Mc} \vec{L}$$

Zeeman effect

Classically, $E = -\vec{\mu} \cdot \vec{B}$

$$\Rightarrow \quad \hat{H} = -\vec{\mu} \cdot \vec{B}$$

Say $\vec{B} = B\hat{z}$ (constant field)

$$\hat{H} = -\frac{qB}{2Mc} \hat{L}_z$$

magnetic quantum #

$$\hat{H} Y_{\ell m} = -\frac{q\hbar m}{2Mc} B \, Y_{\ell m}$$

mass

$$E_m = -\frac{q\hbar B}{2Mc} m, \quad m = -\ell, \dots, \ell$$

For electrons, $q = -e$

$$E_m = \mu_B B m, \qquad m = -l, \ldots, l$$

$$\mu_B = \frac{e\hbar}{2Mc} = \text{Bohr magneton}$$

The Zeeman effect lifts the degeneracy associated with different m-values:

Energy splitting

$$\Delta E = \frac{q \hbar B}{2Mc}$$

To resolve this splitting, must measure for time Δt:

$$\Delta E \, \Delta t \geq \hbar/2$$

$$\Delta t \geq \frac{\hbar/2}{q\hbar B/2mc} = \frac{mc}{qB}$$

Classically, a magnetic field exerts torque on a magnetic moment:

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \frac{d\vec{L}}{dt}$$

Quantum mechanically, this equation of motion still holds on average:

$$\frac{d}{dt}\langle \vec{L} \rangle = \langle \vec{\mu} \rangle \times \vec{B}$$

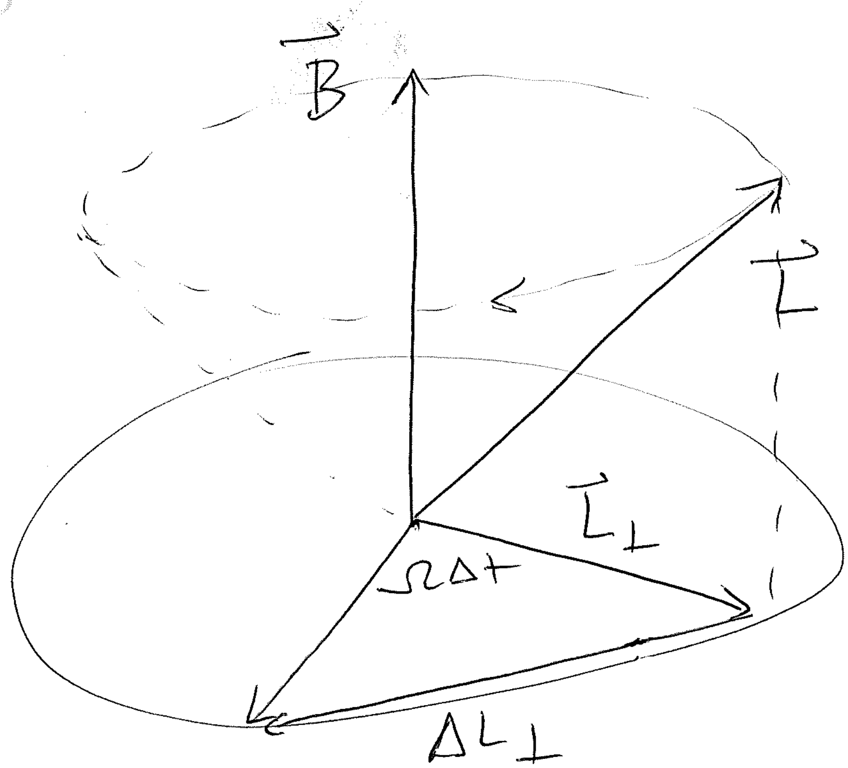
HW: Prove this!

$$\frac{d}{dt}\langle \vec{L} \rangle = -\frac{q\vec{B}}{2mc} \times \langle \vec{L} \rangle \equiv \vec{\Omega} \times \langle \vec{L} \rangle$$

The average angular momentum vector precesses about the magnetic field with angular frequency

$$\boxed{\Omega = |\vec{\Omega}| = \frac{qB}{2Mc} \qquad \text{Larmor frequency}}$$

$$\Omega \Delta t \geq \frac{qB}{2Mc}\frac{Mc}{qB} = \frac{1}{2}$$



$$\frac{\Delta L_\perp}{L_\perp} \sim \frac{1}{2}$$

Thus, applying a magnetic field in the z-direction for a time $> \Delta t$ allows one to resolve the different Zeeman split levels, and hence to measure the z-component of the angular momentum. However, such a measurement necessarilly perturbs L_x and L_y, consistent with the uncertainty relation

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle|.$$

$$\frac{d\langle\vec{L}\rangle}{dt} = \vec{\Omega} \times \langle\vec{L}\rangle$$

$$\vec{\Omega} = -\frac{q\vec{B}}{2mc} = \underbrace{\frac{eB}{2mc}}_{\Omega}\hat{z}$$

Ω = Larmor frequency

(Derivable from

$$\frac{d\langle\vec{L}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{\vec{L}}, \hat{H}]\rangle)$$

$$\vec{\Omega} \times \langle\vec{L}\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ \langle L_x\rangle & \langle L_y\rangle & \langle L_z\rangle \end{vmatrix}$$

$$= -\hat{i}\,\Omega\langle L_y\rangle + \hat{j}\,\Omega\langle L_x\rangle$$

$$\frac{d\langle L_x\rangle}{dt} = -\Omega\langle L_y\rangle$$

$$\frac{d\langle L_y\rangle}{dt} = \Omega\langle L_x\rangle$$

$$\Delta \langle L_x \rangle \simeq -\Omega \Delta t \langle L_y \rangle$$

$$\Delta \langle L_y \rangle \simeq \Omega \Delta t \langle L_x \rangle$$

Uncertainty caused by perturbation:

$$\Delta L_x = |\Delta \langle L_x \rangle| = \Omega \Delta t \, |\langle L_y \rangle|$$

$$\Delta E = \Delta \hat{H}_z = \gamma B \Delta L_z = \Omega \Delta L_z$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$ energy-time uncertainty

$$\Omega \Delta L_z \geq \frac{\hbar}{2\Delta t}, \quad \Delta L_z \geq \frac{\hbar}{2\Omega \Delta t}$$

$$\Delta L_x \Delta L_z \geq \Omega \Delta t \, |\langle L_y \rangle| \frac{\hbar}{2\Omega \Delta t} = \frac{\hbar}{2} |\langle L_y \rangle|$$

same as the result of the generalized uncertainty principle:

$$\Delta L_x \Delta L_z \geq \frac{1}{2} |\langle [\hat{L}_x, \hat{L}_z] \rangle| = \frac{\hbar}{2} |\langle L_y \rangle|$$

Intrinsic magnetic moment

There is also a magnetic moment associated with the spin of the electron. However, because $\vec{S} \neq \vec{r} \times \vec{p}$, the derivation $\vec{\mu} = -\frac{e}{2mc}\vec{S}$ does not work. Instead,

$$\vec{\mu} = g\left(-\frac{e}{2mc}\right)\vec{S},$$

where $g \simeq 2$ (this factor comes from relativistic quantum mechanics, and is one of the most precisely known quantities in the physical sciences).

The Zeeman effect for the spin

is thus (for $\vec{B} = B\hat{z}$ = const.)

$$-\vec{\mu}\cdot\vec{B} = \frac{geB}{2Mc} S_z = \pm \frac{g}{2}\frac{e\hbar}{2Mc} B$$

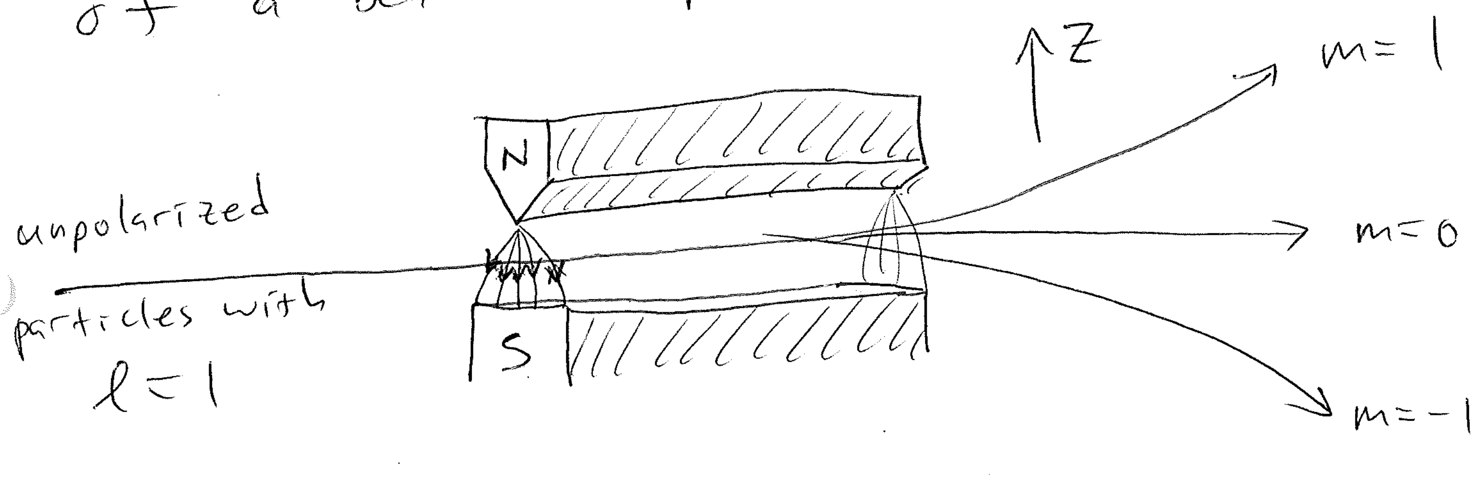
$$= \pm \frac{g}{2} \mu_B B$$

Stern-Gerlach experiment

In an inhomogeneous magnetic field, there is not only a torque, but also a force on a magnetic moment

$$\vec{F} = \nabla(\vec{\mu}\cdot\vec{B}).$$

This force can be used to separate out the different m-components of a beam of particles



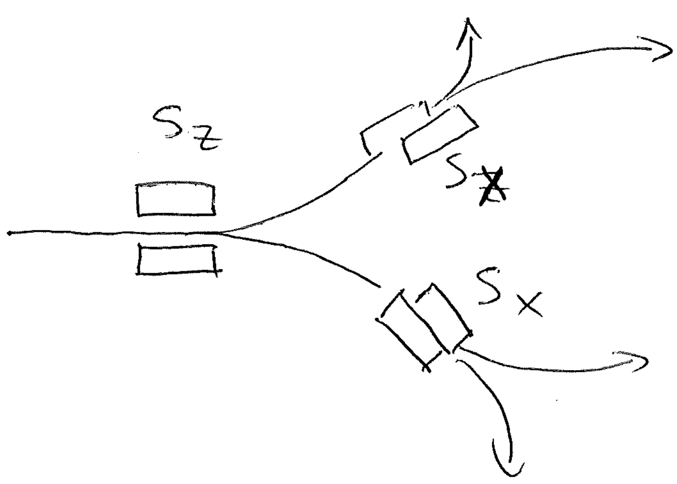
For an unpolarized beam of particles, the # of components in ~~#~~ the beam downstream of the magnet determines the total angular momentum quantum #: $N = 2\ell + 1$.

For an individual particle, the deflection represents a measurement (determination) of the z-component of the angular momentum.

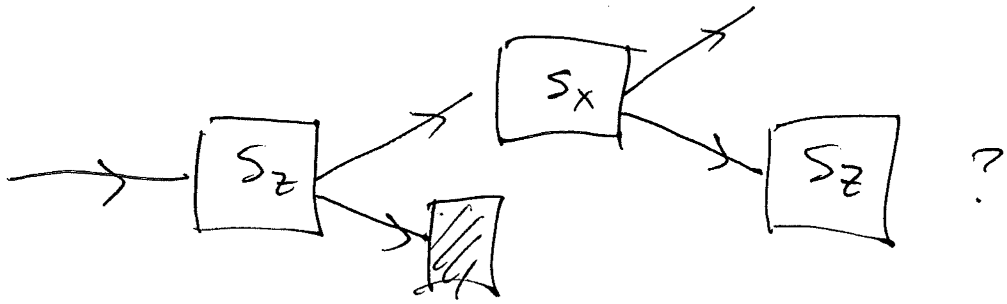
Let's focus on the simplest case, $s = 1/2$. It would be difficult to do the experiment with a free electron, because the beam would be bent and dispersed by the magnetic field, due to the Lorentz force. But for a neutral particle, such as a hydrogen atom, the charge of the electron is balanced by that of the proton, but the magnetic moment is not since

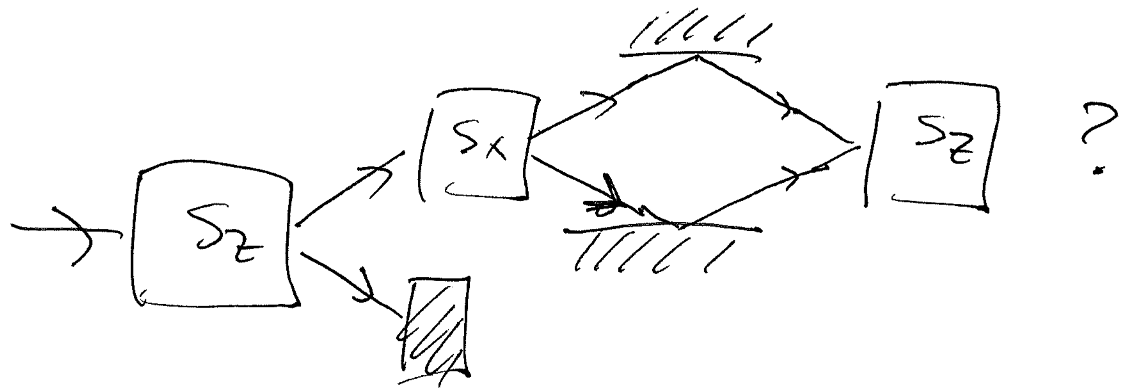
$$\mu_B = \frac{e\hbar}{2m_e c} \gg \mu_p \sim \frac{e\hbar}{2m_p c}$$

Q: What happens if we follow a measurement of S_z by a measurement of S_x?

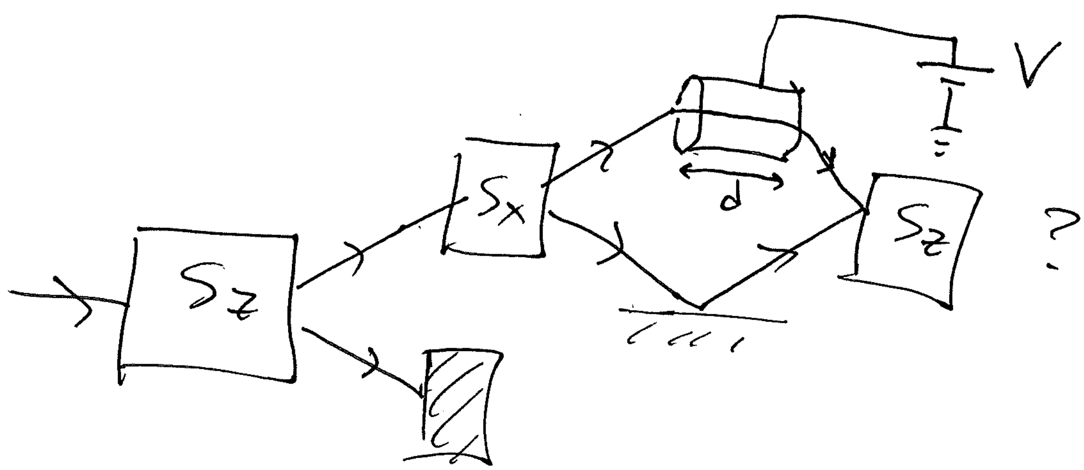


Each eigenstate of S_z is an equal superposition of S_x eigenstates.





Schematics of 2 triple measurements



A modified triple measurement