Physics 570A lecture 29

Addition of angular momenta

Consider two spin-1/2 particles, say the proton and neutron inside a deuteron. What can we say about the total angular momentum, assuming the orbital angular momentum is zero?

Let $\vec{S}_1$ and $\vec{S}_2$ be the spin operators for the two particles. $[\vec{S}_1, \vec{S}_2] = 0$ since they refer to different particles.

Since there are two linearly independent states for each spin, there are four linearly indep. states for two spins. Choosing the z-axis as the axis of the quantization, we can write these states as

(1) $|\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle$

where the first symbol refers to the the eigenvalue of S_{1z} and the second to the eigenvalue of S_{2z}.

The total spin angular momentum of the two particles is

$$\vec{S} = \vec{S}_1 + \vec{S}_2 .$$

$\vec{S}$ obeys canonical commutation

relations for angular momentum:

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}]$$

$$= [S_{1x}, S_{1y}] + [S_{2x}, S_{2y}]$$

$$= i\hbar (S_{1z} + S_{2z}) = i\hbar S_z,$$

etc. Thus the eigenstates of $\vec{S}$ may be chosen as follows:

$$\vec{S}^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle$$

$$S_z |s, m\rangle = \hbar m |s, m\rangle$$

$$S_\pm |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)}\, |s, m \pm 1\rangle$$

with $-s \leq m \leq s$. ($2s+1$ values)

s may be an integer or half-odd integer. The allowed values of m differ by integers.

The basis states (1) are eigenstates of S_z:

$$S_z|\uparrow\uparrow\rangle = \left(\frac{\hbar}{2}+\frac{\hbar}{2}\right)|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$$

$$S_z|\uparrow\downarrow\rangle = \left(\frac{\hbar}{2}-\frac{\hbar}{2}\right)|\uparrow\downarrow\rangle = 0$$

$$S_z|\downarrow\uparrow\rangle = \left(-\frac{\hbar}{2}+\frac{\hbar}{2}\right)|\downarrow\uparrow\rangle = 0$$

$$S_z|\downarrow\downarrow\rangle = \left(-\frac{\hbar}{2}-\frac{\hbar}{2}\right)|\downarrow\downarrow\rangle = -\hbar|\downarrow\downarrow\rangle$$

The eigenvalues of S_z for the system of two particles are clearly 0 and $\pm\hbar$. We expect then to find a triplet of states with $S_z = -\hbar, 0, +\hbar$, corresponding to $S=1$ and a singlet with $S_z=0$, corresponding to $S=0$.

The state $|\uparrow\uparrow\rangle$ cannot have $S=0$, therefore it must have $S=1$, i.e.,

$$\vec{S}^2|\uparrow\uparrow\rangle = \hbar^2 S(S+1)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle.$$

To prove ~~t~~his, note that

$$\vec{S}^2 = (\vec{S}_1+\vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$= \frac{3}{2}\hbar^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+},$$

where

$$S_{1\pm} = S_{1x} \pm i S_{1y},$$

$$S_{2\pm} = S_{2x} \pm i S_{2y}.$$

$$\Rightarrow \vec{S}^2|\uparrow\uparrow\rangle = \left(\frac{3}{2}\hbar^2 + 2\frac{\hbar^2}{4}\right)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$

since the terms involving S_{1+} or S_{2+} give zero.

By the same argument, the state $|\downarrow\downarrow\rangle$ is also a state with $s=1$. To construct the state with $s=1$, $m=0$, we can act upon $|\uparrow\uparrow\rangle$ with S_-:

$$S_-|\uparrow\uparrow\rangle = S_-|1,1\rangle = \hbar\sqrt{2-1\cdot 0}\,|1,0\rangle$$
$$= \hbar\sqrt{2}\,|1,0\rangle$$

But $S_- = S_{1-} + S_{2-}$, so

$$S_-|\uparrow\uparrow\rangle = S_{1-}|\uparrow\uparrow\rangle + S_{2-}|\uparrow\uparrow\rangle$$
$$= \hbar|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle$$

$$\boxed{|1,0\rangle = \frac{S_-}{\hbar\sqrt{2}}|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}$$

The state with $S=0$, $m=0$ must be orthogonal to $|1,0\rangle$,

so

$$\boxed{|s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}.$$

singlet

The three states with $S=1$ are

$$|s=1, m=1\rangle = |\uparrow\uparrow\rangle$$

$$|s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|s=1, m=-1\rangle = |\downarrow\downarrow\rangle$$

triplet

→ deuteron in triplet spin state

Two spin-1/2's added together give total angular momentum $S=1$ or $S=0$. One can think of the spins as being "parallel" in the $S=1$ state and "antiparallel" in the $S=0$ state:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right) = \frac{1}{2}\vec{S}^2 - \frac{3}{4}\hbar^2$$

$$\vec{S}_1 \cdot \vec{S}_2 \, |S=1, m\rangle = \left(\frac{1}{2} 2\hbar^2 - \frac{3}{4}\hbar^2\right) |S=1, m\rangle$$

$$= \frac{1}{4}\hbar^2 \, |S=1, m\rangle$$

$$\vec{S}_1 \cdot \vec{S}_2 \, |S=0, m=0\rangle = -\frac{3}{4}\hbar^2 \, |S=0, m=0\rangle$$

Note that the singlet spin wavefunction is antisymmetric under interchange of particles, while the three triplet spin wavefunctions are symmetric under interchange of particles.

Example: Ortho + para hydrogen.

Addition of angular momenta II

1) Spin-orbit coupling

$$\vec{J} = \vec{L} + \vec{S}$$

$$[\vec{L}, \vec{S}] = 0$$

$$\Rightarrow \quad [J_x, J_y] = i\hbar J_z \quad , \text{ etc.}$$

$$[\vec{S}^2, \vec{J}] = 0 \quad , \quad [\vec{L}^2, \vec{J}] = 0$$

but $\quad [S_z, \vec{J}^2] \neq 0 \qquad [L_z, \vec{J}^2] \neq 0$

$$[J_z, \vec{J}^2] = [L_z + S_z, \vec{J}^2] = 0$$

$J_z = L_z + S_z$ is a good quantum #.

Eigenstates of total ang. momentum:

$$\vec{J}^2 |j m_j\rangle = \hbar^2 j(j+1) |j m_j\rangle$$

$$J_z |j m_j\rangle = \hbar m |j m_j\rangle$$

$$J_\pm |j m_j\rangle = \hbar \sqrt{j(j+1) - m_j(m_j \pm 1)} \, |j \, m_j \pm 1\rangle$$

$$J_\pm = J_x \pm i J_y$$

The states $|j m_j\rangle$ are linear combinations of the states

$$|l m_l\rangle |s m_s\rangle.$$

2) $\underbrace{\text{General problem: addition of}}$ $\vec{J}_1 + \vec{J}_2$

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \quad , \quad [\vec{J}_1, \vec{J}_2] = 0$$

For given quantum #s j_1 & j_2, there are $(2j_1+1)(2j_2+1)$ basis states of the form

$$|j_1 m_1\rangle |j_2 m_2\rangle \equiv |j_1 m_1 j_2 m_2\rangle$$

We would like to find the eigenstates of $\vec{J}$, which are linear combinations of these basis states. Because

$$[\vec{J}_1^2, \vec{J}] = 0 \quad \text{and} \quad [\vec{J}_2^2, \vec{J}] = 0,$$

the eigenstates of $\vec{J}$ are also eigenstates of $\vec{J}_1^2$, $\vec{J}_2^2$, and we may label them

$$|j_1 j_2 j m\rangle.$$

Multiplying by the unit operator in this subspace gives

$$|j_1 j_2 j m\rangle = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle$$
$$\times |j_1 m_1 j_2 m_2\rangle$$

The <u>Clebsch-Gordan coefficients</u>

$$\langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle \equiv \langle j_1 m_1 j_2 m_2 | j m\rangle$$

give the desired linear combination.

Note that

$$\langle j_1 m_1 j_2 m_2 | J_z | j m\rangle = \hbar m \langle j_1 m_1 j_2 m_2 | j m\rangle$$
$$= \hbar (m_1 + m_2) \langle j_1 m_1 j_2 m_2 | j m\rangle$$

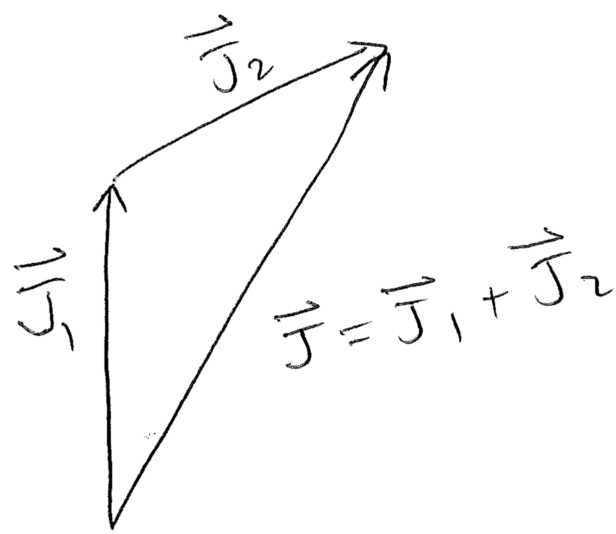
$\Rightarrow \langle j_1 m_1 j_2 m_2 | j m\rangle = 0$ unless

$$m = m_1 + m_2$$

$$|j m\rangle = \sum_{m_1 + m_2 = m} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2 | j m\rangle$$

Another important property of the Clebsch-Gordon coefficients is that

$$\boxed{\begin{array}{l} \langle j_1 m_1 j_2 m_2 | j m \rangle \neq 0 \quad \text{only if} \\ |j_1 - j_2| \leq j \leq j_1 + j_2 \\ \underline{\text{triangle inequality}} \end{array}}$$



For a proof of the triangle rule, consider the possible values of m.

Note that $\max\{m\} = j_1 + j_2$.

Thus $\max\{j\} = j_1 + j_2$;
otherwise a higher value of
m could be generated by
acting upon the state with
$m = j_1 + j_2$ with J_+.

If $j_1 + j_2 \in \mathbb{Z}$ (integers),
then $m_1 + m_2 \in \mathbb{Z}$, and $m = m_1 + m_2 \in \mathbb{Z}$
so $j \in \mathbb{Z}$. On the other hand,
if $j_1 + j_2$ is a half-odd integer,
e.g. $j_1 = 1$, $j_2 = 1/2$, then
$m = m_1 + m_2$ is a half-odd integer,
and j is a half-odd integer.
In either case, the allowed
values of j and m differ by

integers.

To determine the minimum value of $j = |j_1 - j_2|$, one can use an inductive argument, as discussed in class. However, one can also use the <u>Schwartz inequality</u>:

$$-\sqrt{\langle \vec{J}_1^{\,2}\rangle\langle \vec{J}_2^{\,2}\rangle} \leq \langle \vec{J}_1\cdot\vec{J}_2\rangle \leq \sqrt{\langle \vec{J}_1^{\,2}\rangle\langle \vec{J}_2^{\,2}\rangle}$$

$$\langle \vec{J}_1\cdot\vec{J}_2\rangle = \frac{1}{2}\langle \vec{J}^{\,2} - \vec{J}_1^{\,2} - \vec{J}_2^{\,2}\rangle$$

$$= \frac{\hbar^2}{2}\left(J(J+1) - J_1(J_1+1) - J_2(J_2+1)\right)$$

$$\Rightarrow \boxed{J(J+1) \geq J_1(J_1+1) + J_2(J_2+1) - 2\sqrt{J_1(J_1+1)J_2(J_2+1)}}$$

Suppose, without loss of generality, 8

that $J_1 \geq J_2$.

i) Check if $J = J_1 - J_2$ satisfies the inequality.

$$J(J+1) = (J_1 - J_2)(J_1 - J_2 + 1)$$

$$= J_1(J_1+1) + J_2(J_2+1) - 2J_2(J_1+1)$$

$$-J_2(J_1+1) \overset{?}{\geq} -\sqrt{J_1(J_1+1)J_2(J_2+1)}$$

$$J_2^2(J_1+1)^2 \leq J_1(J_1+1)J_2(J_2+1)$$

$$J_2(J_1+1) \leq J_1(J_2+1)$$

$$J_1J_2 + J_2 \leq J_1J_2 + J_1$$

$$J_2 \leq J_1$$

ii) Check $\underline{J = J_1 - J_2 - 1}$

$$J(J+1) = (J_1 - J_2 - 1)(J_1 - J_2)$$
$$= J_1(J_1+1) + J_2(J_2+1) - 2J_1(J_2+1)$$

$$-J_1(J_2+1) \overset{?}{\geq} -\sqrt{J_1(J_1+1)J_2(J_2+1)}$$

$$J_1^2(J_2+1)^2 \leq J_1(J_1+1)J_2(J_2+1)$$

$$J_1(J_2+1) \leq (J_1+1)J_2$$

$$J_1J_2 + J_1 \leq J_1J_2 + J_2$$

$$J_1 \leq J_2 \Rightarrow \text{contradiction}$$

Therefore

$$|j_1 - j_2| \leq j \leq j_1 + j_2 .$$

As a final test, let us verify the size of the $|j_1 j_2 j m\rangle$ Hilbert space:

$$N_H = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1)$$

$$= 2(j_1+j_2)+1 + \cdots + 2(j_1-j_2)+1$$

$$= \left[j_1+j_2-(j_1-j_2)+1\right] \times \frac{1}{2}\left[2(j_1+j_2)+1 + 2(j_1-j_2)+1\right]$$

$$= (2j_2+1)(2j_1+1)$$

This is the same # of states as in the $|j_1 m_1\rangle |j_2 m_2\rangle$ basis.

Physics 570A Lecture 29.2

Addition of angular momenta III

Recall $|jm\rangle = \sum_{m_1+m_2=m} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2 | jm\rangle$,

where $\langle j_1 m_1 j_2 m_2 | jm\rangle$ is a Clebsch-Gordon coefficient. Taking the inner product with $\langle jm|$ gives

$$\underline{1} = \sum_{m_1+m_2=m} |\langle j_1 m_1 j_2 m_2 | jm\rangle|^2 .$$

The sum of the squares of the Clebsch-Gordon coefficients is one.

1) $\underline{J_1 + S = 1/2}$

An important case, and a simple one, is the addition of $S=1/2$ to another angular momentum J_1.

A trivial case is $J_1 = 0$.

Then $\vec{J} = \vec{J}_1 + \vec{S}$, with $J = 1/2$.

For $J_1 \equiv l \neq 0$, there are exactly two allowed values

$$J = l \pm 1/2.$$

Let us construct the corresponding Clebsch-Gordon coefficients.

$$|J\ m\rangle = |l+1/2, l+1/2\rangle = |l\ l\ 1/2\ 1/2\rangle$$

$$|l\ m_l\ s\ m_s\rangle$$

(Clebsch-Gordon coeff. = 1)

$$J_-|l+1/2, l+1/2\rangle = \hbar\sqrt{(l+1/2)(l+3/2)-(l+1/2)(l-1/2)}$$

$$\times |l+1/2, l-1/2\rangle$$

$$= \hbar\sqrt{2l+1}\ |l+1/2, l-1/2\rangle$$

$$J_- |l\, l\, \tfrac{1}{2}\, \tfrac{1}{2}\rangle = J_{1-} |l\, l\rangle |\tfrac{1}{2}\, \tfrac{1}{2}\rangle$$
$$+ |l\, l\rangle J_{2-} |\tfrac{1}{2}\, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{l(l+1) - l(l-1)}\, |l\, l-1\rangle |\tfrac{1}{2}\, \tfrac{1}{2}\rangle$$
$$+ \hbar\, |l\, l\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2l}\, |l\, l-1\, \tfrac{1}{2}\, \tfrac{1}{2}\rangle + \hbar\, |l\, l\, \tfrac{1}{2}\, -\tfrac{1}{2}\rangle$$

$$\Rightarrow \quad |l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \sqrt{\frac{2l}{2l+1}}\, |l\, l-1\, \tfrac{1}{2}\, \tfrac{1}{2}\rangle$$
$$+ \sqrt{\frac{1}{2l+1}}\, |l\, l\, \tfrac{1}{2}\, -\tfrac{1}{2}\rangle$$

But we can also construct a linear combination of $|l\, l-1\, \tfrac{1}{2}\, \tfrac{1}{2}\rangle$ and $|l\, l\, \tfrac{1}{2}\, -\tfrac{1}{2}\rangle$ which is orthogonal to $|l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle$. This can only be the state $|l-\tfrac{1}{2}, l-\tfrac{1}{2}\rangle$.

$$|l-1/2, l-1/2\rangle = \sqrt{\frac{1}{2l+1}}\, |l\ l-1\ 1/2\ 1/2\rangle$$
$$- \sqrt{\frac{2l}{2l+1}}\, |l\ l\ 1/2\ -1/2\rangle$$

We can repeat the process to generate the states with lower values of total m. The result is

$$\langle l\ m \mp 1/2\ \ 1/2\ \pm 1/2 | l + 1/2\ m\rangle = \sqrt{\frac{l \pm m + 1/2}{2l+1}}$$

$$\langle l\ m \mp 1/2\ \ 1/2\ \pm 1/2 | l - 1/2\ m\rangle = \pm\sqrt{\frac{l \mp m + 1/2}{2l+1}}$$

The values of the Clebsch-Gordon coefficients for other values of J_1 and J_2 have been tabulated (see Griffiths for refs.).

2) Spin-Orbit interaction

Consider an electron traveling at velocity $\vec{v}$ through an electric field $\vec{E}$. In the rest frame of the electron, there is a magnetic field $\vec{B}' = -\frac{\vec{v}}{c} \times \vec{E}$.

This field will couple to the magnetic moment, leading to a correction to the Hamiltonian of the form

$$-\vec{\mu} \cdot \vec{B}', \quad \text{where} \quad \vec{\mu} = g \frac{(-e)}{2m_e c} \vec{S}$$

and $g \simeq 2$ for an electron in vacuum. There is a further relativistic correction due to the Thomas precession, which reduces the effect by a factor of two :

$$H_{s.o.} = -g \frac{e}{4m_e c^2} \vec{S} \cdot (\vec{v} \times \vec{E})$$

If the electric field is due to a central potential $V(r)$, as occurs to a first approximation in an atom, then

$$e\vec{E} = \nabla V = \frac{\vec{r}}{r} \frac{dV}{dr} \qquad \text{and}$$

$$H_{s.o.} = \frac{g}{4m_e c^2} \vec{S} \cdot (\vec{r} \times \vec{v}) \frac{1}{r} \frac{dV}{dr}$$

$$= \left(\frac{g}{4m_e^2 c^2} \frac{1}{r} \frac{dV}{dr} \right) \vec{L} \cdot \vec{S}$$

The eigenstates of $\vec{J} = \vec{L} + \vec{S}$ are eigenstates of $\vec{L}\cdot\vec{S}$:

$$\vec{L}\cdot\vec{S}\,|l s j m\rangle = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)|l s j m\rangle$$

$$= \frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)|l s j m\rangle$$

i) For $j = l + 1/2$, we have

$$\frac{\vec{L}\cdot\vec{S}}{\hbar^2}|l s j m\rangle = \frac{l}{2}|l s j m\rangle$$

ii) For $j = l - 1/2$, we have

$$\frac{\vec{L}\cdot\vec{S}}{\hbar^2}|l s j m\rangle = -\frac{l-1}{2}\,|l s j m\rangle$$

The eigenvalue is positive for $\vec{L}$ parallel to $\vec{S}$ and negative for $\vec{L}$ antiparallel to $\vec{S}$.