Phys 570A Midterm 1
Solutions

1) Switch to matrix notation:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \qquad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where $H_{ij} = \langle i | \hat{H} | j \rangle$, $C_{ij} = \langle i | \hat{C} | j \rangle$

$$H = \begin{pmatrix} \varepsilon & V \\ V & \varepsilon \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

a) Energy eigenvalues

$$0 = \begin{vmatrix} \varepsilon - E & V \\ V & \varepsilon - E \end{vmatrix} = (\varepsilon - E)^2 - V^2$$

$$E_{\pm} = \varepsilon \pm V$$

Eigenstates

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle \mp |2\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}$$

Check: $|\pm\rangle = \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix}$

$$(\mathcal{E} - E_\pm)\alpha_\pm + V\beta_\pm = 0$$

$$\beta_\pm = \frac{\mathcal{E} - E_\pm}{V}\alpha_\pm = \mp \alpha_\pm \quad \checkmark$$

$$|\psi(0)\rangle = |1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|+\rangle e^{-\frac{iE_+ t}{\hbar}} + \frac{1}{\sqrt{2}}|-\rangle e^{-\frac{iE_- t}{\hbar}}$$

Outcomes: $E_\pm = \mathcal{E} \pm V$

Probabilities: $\mathcal{P}(E_\pm) = \frac{1}{2}$

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b) Possible outcomes: $C = 1, 2$

$$\mathcal{P}(C=1) = |\langle 1|\psi(t)\rangle|^2$$

$$\mathcal{P}(C=2) = |\langle 2|\psi(t)\rangle|^2$$

$$\langle 1|\psi(t)\rangle = \frac{1}{\sqrt{2}}\langle 1|+\rangle e^{\frac{-iE_+t}{\hbar}} + \frac{1}{\sqrt{2}}\langle 1|-\rangle e^{\frac{-iE_-t}{\hbar}}$$

$$= \frac{1}{2} e^{\frac{-i\varepsilon t}{\hbar}}\left(e^{\frac{-iVt}{\hbar}} + e^{\frac{iVt}{\hbar}}\right)$$

$$P(C=1) = \cos^2\left(\frac{Vt}{\hbar}\right)$$

$$\langle 2|\psi(t)\rangle = \frac{1}{\sqrt{2}}\langle 2|+\rangle e^{\frac{-iE_+t}{\hbar}} + \frac{1}{\sqrt{2}}\langle 2|-\rangle e^{\frac{-iE_-t}{\hbar}}$$

$$= -\frac{1}{2} e^{\frac{-iE_+t}{\hbar}} + \frac{1}{2} e^{\frac{-iE_-t}{\hbar}}$$

$$P(C=2) = \sin^2\left(\frac{Vt}{\hbar}\right)$$

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c) $\Delta E\, \Delta C \geq \frac{1}{2}\left|\langle[\hat{C},\hat{H}]\rangle\right|$

$$[\hat{C},\hat{H}] = -\begin{pmatrix}\varepsilon & V\\ V & \varepsilon\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\varepsilon & V\\ V & \varepsilon\end{pmatrix}$$

$$= \begin{pmatrix}-\varepsilon & -2V\\ -V & -2\varepsilon\end{pmatrix} + \begin{pmatrix}\varepsilon & V\\ 2V & 2\varepsilon\end{pmatrix} = \begin{pmatrix}0 & -V\\ V & 0\end{pmatrix}$$

For a general state

$$|\psi\rangle = \frac{\alpha|1\rangle + \beta|2\rangle}{\sqrt{|\alpha|^2+|\beta|^2}},$$

$$\langle [\hat{C}, \hat{H}] \rangle = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} 0 & -V \\ V & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \frac{1}{|\alpha|^2 + |\beta|^2} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} -V\beta \\ V\alpha \end{pmatrix} = \frac{V(\alpha\beta^* - \alpha^*\beta)}{|\alpha|^2 + |\beta|^2}$$

$$\Delta E \Delta C \geq \frac{|V| \, |\mathrm{Im}\, \alpha\beta^*|}{|\alpha|^2 + |\beta|^2}$$

Lower bound is zero if $\alpha = 0$ or $\beta = 0$ or $\mathrm{Im}\, \alpha\beta^* = 0$, e.g.

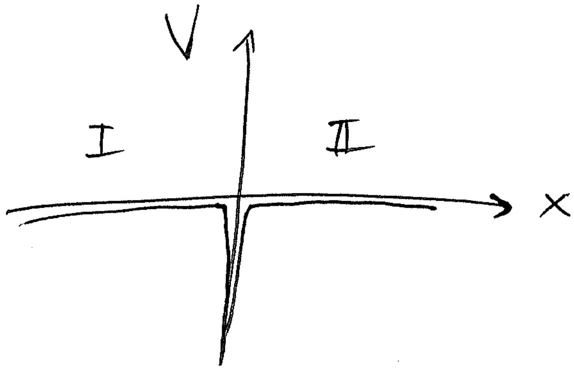
$|\psi\rangle = |1\rangle$ or $|\psi\rangle = |2\rangle$ or

$$|\psi\rangle = e^{i\theta} \frac{|1\rangle \pm |2\rangle}{\sqrt{2}}$$

(Eigenstates of $\hat{H}$ and $\hat{C}$.)

2) $V(x) = -\lambda \delta(x)$

a)



$\psi_I(x) = A e^{Kx}$

$\psi_{II}(x) = B e^{-Kx}$

i) $\psi_I(0) = \psi_{II}(0) \Rightarrow A = B$

ii) $-\frac{\hbar^2}{2m}\left(\psi_{II}'(0) - \psi_I'(0)\right) - \lambda\psi(0) = 0$

$\psi_{II}'(0) - \psi_I'(0) = -\frac{2m\lambda}{\hbar^2}\psi(0)$

$(-K - K)A = \frac{-2m\lambda}{\hbar^2} A$

$\Rightarrow K = \frac{m\lambda}{\hbar^2}$

$$\boxed{E_0 = -\frac{\hbar^2 K^2}{2m} = -\frac{m\lambda^2}{2\hbar^2}}$$

b) $\psi_I = e^{ikx} + r e^{-ikx}$

$\psi_{II} = t e^{ikx}$

$E = \frac{\hbar^2 k^2}{2m}$

i) $\psi_I(0) = \psi_{II}(0) \implies 1 + r = t$

ii) $\psi_{II}'(0) - \psi_I'(0) = -\frac{2m\lambda}{\hbar^2}\psi(0)$

$$ik(t - (1-r)) = -\frac{2m\lambda}{\hbar^2} t$$

$$t + r - 1 = \frac{2im\lambda}{\hbar^2 k} t$$

$$2t - 2 = \frac{2im\lambda}{\hbar^2 k} t$$

$$t\left(1 - \frac{im\lambda}{\hbar^2 k}\right) = 1 \qquad \text{Let } \frac{\hbar^2}{m\lambda} = l$$

$$t = \frac{1}{1 - \frac{im\lambda}{\hbar^2 k}} = \frac{1}{1 - \frac{i}{kl}} = \frac{ikl}{ikl + 1}$$

$$\boxed{T = |t|^2 \frac{k_{II}}{k_I} = |t|^2 = \frac{k^2 l^2}{k^2 l^2 + 1} \qquad T(E) = \frac{E/|E_0|}{E/|E_0| + 1}}$$

3) $$\frac{d\langle Q\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{Q},\hat{H}]\rangle$$

$$\Delta E\,\Delta Q \geq \frac{1}{2}\left|\langle[\hat{Q},\hat{H}]\rangle\right| = \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$$

$$\Delta E\,\frac{\Delta Q}{\left|\frac{d\langle Q\rangle}{dt}\right|} \geq \frac{\hbar}{2}$$

$$\Delta E\,\Delta t \geq \frac{\hbar}{2}$$